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Phase transitions in thermoelastic and thermoviscoelastic shells

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Applying the general non-linear theory of shells undergoing phase transitions, we derive the balance equations along the singular surface curve modelling the phase interface in the shell. From the integral forms of balance laws of linear momentum, angular momentum, and energy as well as the entropy inequality, we obtain the local static balance equations along the curvilinear phase interface. We discuss general forms of the constitutive equations for thermoelastic and thermoviscoelastic shells, as well as propose their simple cases for the linear isotropic shell behaviour. We also derive the thermodynamic condition allowing one to determine the interface position on the deformed shell midsurface. The theoretical model is illustrated by the example of thin circular cylindrical shell made of a two-phase elastic material subjected to tensile forces at the shell boundary. The solution reveals the existence of the hysteresis loop whose size depends upon values of several loading parameters.

Key words: non-linear shell, phase transition, kinetic equation, quasi-static loading, thermoviscoelasticity, extended cylinder.

1. Introduction

Smart materials are widely used in modern technology. Examples of such materials are shape memory alloys and shape memory polymers [45]. The diffusionless (displacive) phase transitions (PT) of martensitic type are recognised [7, 28] to be responsible for the shape memory effect in these materials. Phase transitions in steel, alloys, polymers and other materials become very important also for contemporary materials science.

Within continuum mechanics the PT phenomena can be modelled using different approaches. The approach used in this paper is based on introducing the sharp phase interface, being a sufficiently smooth surface dividing two different material phases. In this approach the position of the phase interface itself becomes unknown, together with the translation, temperature and other unknown fields.

Mechanics of two-phase media has been developed in a number of papers and summarized in books, for example by Grinfeld [19], Gurtin [21, 22], Romano [43], Bhattacharya [7], Abeyaratne and Knowles [2], Lagoudas *et al.* [28], and Berezovski *et al.* [5]. Starting from the pioneering work by Gibbs [16], the balance equations at the phase interface and the phase interface motion during the deformation process are among the most discussed issues in this field. Apart from the monographs mentioned above, let us remind here some representative papers on equilibrium and motion of the phase boundary [1–4, 6, 15, 26, 29, 30, 33, 38, 44, 53, 54]. In those works many one-dimensional problems were discussed which also adequately described behaviour of bars, rods and beams made of martensitic materials.

In technology there is a growing interest to understand the behaviour of two-dimensional structures such as thin films, plates and shells made of shape memory alloys and other materials undergoing PT. Thin films made of shape memory alloys are regarded to be very prospective for design of microelectromechanical systems (MEMS). Mechanics of such thin films was discussed, for example, by Bhattacharya and James [8], and James and Rizzoni [23], see also [7]. Let us note that experiments on shape memory alloys are usually performed with thin-walled samples, for example rectangular plates [7, 11, 39, 40] or tubes [14, 31, 50]. Thus, it is important to develop two-dimensional mechanics of thin-walled structural elements made of materials undergoing PT, which is based on the theory of shells.

Equilibrium conditions of elastic thin-walled structures (plates and shells) undergoing PT of martensitic type were formulated by Eremeyev and Pietraszkiewicz [12], and Pietraszkiewicz *et al.* [42], within the dynamically and kinematically exact theory of shells presented in the books [10, 32]. By analogy to the 3D case, as the phase boundary in the shell we have taken in [12, 42] a singular surface curve. Hence, the two-phase shell was regarded as a certain material base surface, consisting of two material phases divided by a sufficiently smooth surface curve. This assumption about existence of such a dividing curve restricts the class of allowable deformations of the shell as 3D body, because not for all types of PT the reduction of 3D body to 2D shell leads also to reduction of the phase dividing surface to the surface interface curve. But in many cases, existence of such a curve was confirmed by experiments made on thin-walled samples, see [7, 11, 14, 39, 40, 50]. As examples of other approaches to model PT in thin-walled elements, we note the papers by Shkutin [46, 47] and the review published in [42].

From experimental data we know that PT depending on strain rates and inelastic effects may considerably influence the stress state of the solid. The aim of this paper is to extend the results [12, 42] by taking into account temperature and viscoelastic effects of the shell material phases.

The paper is organised as follows. In Sec. 2 we remind the notation, the exact equilibrium conditions, the shell strain measures as well as the kinematic compatibility conditions and the jump conditions at the curvilinear phase boundary already discussed in [12]. The 2D resultant counterparts of the 3D energy balance and dissipation inequality, together with the corresponding jump conditions at the phase boundary, are discussed in Secs. 3 and 4. Then using the jump conditions we discuss in Sec. 5 the possible forms of the kinetic equation, allowing one to describe motion of the curvilinear phase boundary in the quasistatic deformation process. The initial-boundary value problem is completed by the constitutive equations of thermoelastic and thermoviscoelastic shells whose general structure is discussed in Secs. 6 and 7, where also their simple expressions for the linear isotropic shell behaviour are proposed. Finally, in Sec. 8 we illustrate the theoretical model developed here by an axisymmetric example of a thin circular cylindrical shell made of an elastic material capable of undergoing phase transformations. The cylinder is subjected to tensile forces on the one end and is clumped on another one.

2. Shell equilibrium conditions

A shell is a three-dimensional (3D) solid body which in a reference (undeformed) placement is identified with a region B of the physical space $\mathcal{E}$ with E as its translation 3D vector space. Geometry of B is usually described in the system of normal coordinates $\{\theta^\alpha, \xi\}$, $\alpha = 1, 2$, where $\xi = 0$ defines the shell undeformed base surface $M \subset B$, and $-h^- \leq \xi \leq h^+$ is the distance from M, with $h = h^- + h^+$ denoting the shell thickness. Relative to an inertial frame $(o, \mathbf{i}_k)$, where $o \in \mathcal{E}$ and $\mathbf{i}_k \in E$, $k = 1, 2, 3$, are orthonormal vectors, the position vector $\mathbf{x}$ of an arbitrary point $\mathrm{x} \in B$ is given by

$$\mathbf{x}(\theta^\alpha, \xi) = \boldsymbol{x}(\theta^\alpha) + \xi \boldsymbol{\eta}(\theta^\alpha),$$

where $\boldsymbol{x}(\theta^\alpha) = \mathbf{x}(\theta^\alpha, 0)$ is the position vector of M, $\boldsymbol{\eta} = \dfrac{1}{\sqrt{a}} \boldsymbol{x}_{,1} \times \boldsymbol{x}_{,2}$ is the unit normal vector orienting M, and $a = \det(\boldsymbol{x}_{,\alpha} \cdot \boldsymbol{x}_{,\beta})$, $(\ldots)_{,\alpha} \equiv \dfrac{\partial}{\partial \theta^\alpha}(\ldots)$.

Within the dynamically and kinematically exact theory of shells developed in [10, 41], in the deformed placement the shell is represented by the position vector $\boldsymbol{y} = \chi(\boldsymbol{x})$ of the deformed material base surface $N = \chi(M)$ with attached three directors $(\boldsymbol{d}_\alpha, \boldsymbol{d})$ such that

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}, \quad \boldsymbol{d}_\alpha = \boldsymbol{Q}\boldsymbol{x}_{,\alpha}, \quad \boldsymbol{d} = \boldsymbol{Q}\boldsymbol{\eta}, \tag{2.1}$$

where χ is the deformation function, $\boldsymbol{u} \in E$ is the translation vector of M, and $\boldsymbol{Q} \in SO(3)$ is the proper orthogonal tensor, $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$, $\det \boldsymbol{Q} = +1$, given on

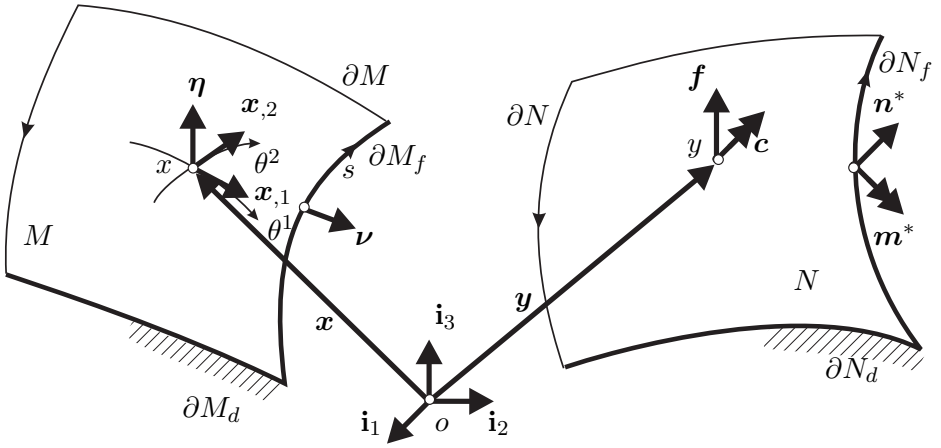

FIG. 1. Shell kinematics.

M representing the work-averaged gross rotation of the shell cross-sections from their undeformed shapes described by $(\boldsymbol{x}_{,\alpha}, \boldsymbol{\eta})$, Fig. 1.

In the shell undergoing phase transition, it is assumed that above some level of deformation different material phases A and B may appear in different complementary subregions N_A and N_B separated by the curvilinear phase interface $D \in N$, Fig. 2. For a piecewise differentiable mapping χ we can introduce on M a singular image curve $C = \chi^{-1}(D)$ separating the corresponding image regions $M_A = \chi^{-1}(N_A)$ and $M_B = \chi^{-1}(N_B)$, Fig. 2. The position vectors of C and D are related by $\boldsymbol{x}_C(s) = \chi^{-1}(\boldsymbol{y}_C(s))$, where s is the arc length parameter along C.

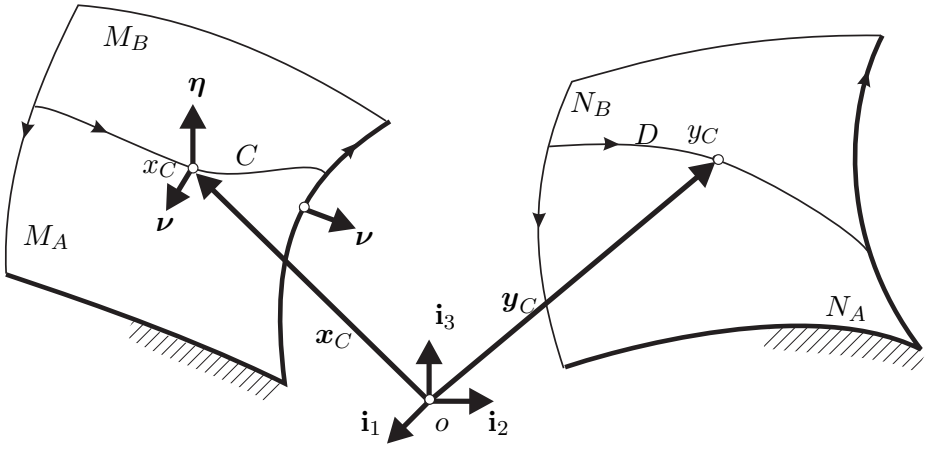

FIG. 2. The shell with phase interface.

The Lagrangian equilibrium BVP for elastic shells with PT was formulated in [12] as the stationary problem for the functional of total potential energy. In the present paper, however, we discuss the thermoviscoelastic shells which allow for some energy dissipation during deformation with PT. But the exact, resultant Lagrangian equilibrium conditions for the shell with the singular surface curve modelling the phase interface, can always be derived by performing direct integration across the shell thickness of the 3D global equilibrium conditions of continuum mechanics, see for example [10, 48, 27].

Let $\boldsymbol{f}(\theta^\alpha)$, $\boldsymbol{c}(\theta^\alpha)$ be the resultant surface force and couple vector fields acting on $N\backslash D$, but measured per unit area of $M\backslash C$, and let $\boldsymbol{n}^\star(s)$, $\boldsymbol{m}^\star(s)$ be the resultant 1D boundary force and couple vector fields acting along ∂N_f, but measured per unit length of ∂M_f. Then the exact, resultant, local Lagrangian equilibrium conditions are [10, 27]

$$\begin{gathered}\operatorname{Div}_s \boldsymbol{N} + \boldsymbol{f} = \boldsymbol{0}, \quad \operatorname{Div}_s \boldsymbol{M} + \operatorname{ax}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T) + \boldsymbol{c} = \boldsymbol{0} \quad \text{in } M\backslash C, \\ \boldsymbol{N}\boldsymbol{\nu} = \boldsymbol{n}^\star, \quad \boldsymbol{M}\boldsymbol{\nu} = \boldsymbol{m}^\star \quad \text{along } \partial M_f\backslash C,\end{gathered} \tag{2.2}$$

where $(\boldsymbol{N}, \boldsymbol{M}) \in E \otimes T_x M$ are the surface tangential stress resultant and stress couple tensors of the first Piola–Kirchhoff type, following from the Cauchy theorem $\boldsymbol{n}_\nu = \boldsymbol{N}\boldsymbol{\nu}$ and $\boldsymbol{m}_\nu = \boldsymbol{M}\boldsymbol{\nu}$ of the resultant contact force $\boldsymbol{n}_\nu$ and couple $\boldsymbol{m}_\nu$ vectors, $\boldsymbol{F} = \operatorname{Grad}_s \boldsymbol{y}$ is the surface deformation gradient, $\boldsymbol{F} \in E \otimes T_x M$, $\operatorname{ax}(\ldots)$ is the axial vector associated with the skew tensor $(\ldots)$, $\boldsymbol{\nu}$ is the surface unit vector externally normal to ∂M, while Grad_s and Div_s are the surface gradient and divergence operators on M as defined in [10, 20], respectively.

Additionally, at the curvilinear phase interface C, which is the singular surface curve with regard to the surface stress measures, we obtain the local Lagrangian equilibrium conditions

$$[\![\boldsymbol{N}\boldsymbol{\nu}]\!] = \boldsymbol{0}, \qquad [\![\boldsymbol{M}\boldsymbol{\nu}]\!] = \boldsymbol{0}, \tag{2.3}$$

where the expression $[\![\ldots]\!] = (\ldots)_B - (\ldots)_A$ means the jump at C.

More general Lagrangian equilibrium conditions along C, taking into account also the elastic strain energy density of the interface itself as well as additional resultant force and couple vectors acting only along C, were discussed in [27, 42].

In the general theory of shells considered here, the following two strain measures corresponding to the deformations (2.1) may be introduced, see [10, 12, 13, 41]:

$$\boldsymbol{E} = \boldsymbol{\varepsilon}_\alpha \otimes \boldsymbol{a}^\alpha, \quad \boldsymbol{K} = \boldsymbol{\varkappa}_\alpha \otimes \boldsymbol{a}^\alpha, \quad \boldsymbol{\varepsilon}_\alpha = \boldsymbol{y}_{,\alpha} - \boldsymbol{d}_\alpha, \quad \boldsymbol{\varkappa}_\alpha = \frac{1}{2}\boldsymbol{d}^i \times \boldsymbol{Q}_{,\alpha}\boldsymbol{Q}^T\boldsymbol{d}_i, \tag{2.4}$$

where $(\boldsymbol{a}^\alpha, \boldsymbol{\eta})$ and $(\boldsymbol{d}^i)$ are the bases reciprocal to $(\boldsymbol{x}_{,\alpha}, \boldsymbol{\eta})$ and $(\boldsymbol{d}_\alpha, \boldsymbol{d})$, respectively.

When deformation is small, one can essentially simplify expressions of the strain measures (2.4). In such a case $\boldsymbol{\varepsilon}_\alpha$ and $\boldsymbol{\varkappa}_\alpha$ are given by, see [10],

$$\boldsymbol{\varepsilon}_\alpha = \boldsymbol{u}_{,\alpha} - \boldsymbol{\varphi} \times \boldsymbol{x}_{,\alpha}, \qquad \boldsymbol{\varkappa}_\alpha = \boldsymbol{\varphi}_{,\alpha}, \tag{2.5}$$

where $\boldsymbol{\varphi}$ is the infinitesimal rotation vector such that $\boldsymbol{Q} \approx \mathbf{1} - \boldsymbol{\varphi} \times \mathbf{1}$ if $\|\boldsymbol{\varphi}\| \ll 1$, and $\mathbf{1}$ is the 3D identity tensor.

Let us consider a one-parameter family of shell deformations

$$\boldsymbol{y}(\boldsymbol{x},t) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x},t), \quad \boldsymbol{d}_\alpha(\boldsymbol{x},t) = \boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{x}_{,\alpha}, \quad \boldsymbol{d}(\boldsymbol{x},t) = \boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{\eta}(\boldsymbol{x}), \tag{2.6}$$

where t is a time-like scalar parameter such that $t = 0$ corresponds to the undeformed placement and t to the deformed one. Then $\boldsymbol{v} = \dot{\boldsymbol{u}}$ is the translational velocity vector, and $\boldsymbol{\omega} = \operatorname{ax}(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T)$ is the angular velocity vector, while $V = \dot{\boldsymbol{x}}_C \cdot \boldsymbol{\nu}$ is the exterior normal velocity of the phase curve C.

The curvilinear phase interfaces in shells can be either coherent or incoherent in rotations, see EREMEYEV and PIETRASZKIEWICZ [12]. For the coherent interface both fields $\boldsymbol{y}$ and $\boldsymbol{Q}$ are supposed to be continuous at C and the kinematic compatibility conditions along C become, see [12], Eqs. (31) and (34),

$$[\![\boldsymbol{v}]\!] + V[\![\boldsymbol{F}\boldsymbol{\nu}]\!] = \mathbf{0}, \qquad [\![\boldsymbol{\omega}]\!] + V[\![\boldsymbol{K}\boldsymbol{\nu}]\!] = \mathbf{0}. \tag{2.7}$$

The phase interface is called incoherent in rotations if only $\boldsymbol{y}$ is continuous at C but the continuity of $\boldsymbol{Q}$ may be violated. In this case the condition $(2.7)_1$ is still satisfied, but $(2.7)_2$ may be violated, [12].

From the physical point of view, the phase interface incoherent in rotations model the singular surface curve at which the position vector remains continuous, but the curvature of the base surface can have a jump discontinuity during deformation process. Such singular surface curves correspond to formation of folds or partial rotational damage of the base surface. In particular, such folds were observed in thin martensitic films as the so-called tents and tunnels, see the experimental data presented in [9, 8, 23, 24]. Of course, the notion of 1D phase interface incoherent in rotations (as used in this paper) is somewhat idealized, for in statics it corresponds to appearance of an ideal hinge associated with the constraint $\boldsymbol{m}_\nu = \mathbf{0}$ along the interface. More realistic models of 1D incoherent phase interface, which would allow creation of folds along which $\boldsymbol{m}_\nu \neq \mathbf{0}$, could be obtained for example by introducing additional constitutive equations for the forces, couples and energies associated with the interface itself, as in [42]. In such a case the interface incoherent in rotations would model a deformable elastic or thermoviscoelastic curvilinear joint. The 2D incoherent interfaces in 3D solids were discussed by GRINFELD [19].

3. Energy balance

To take into account the influence of temperature, let us discuss the balance of energy in the shell. Thermodynamics of shells from various points of view was presented for example in [17, 18, 34, 36, 48, 49, 55]. In the papers various sets of surface fields responsible for temperature were used and several formulations of the first and second laws of thermodynamics for shells were discussed. In order to present here the reasonably simple results we confine ourselves to the simplest version of these laws suggested by MURDOCH [36, 37]. If additionally the temperature changes across the shell thickness are disregarded, at the shell midsurface we have the temperature field T which describes some through-the-thickness average temperature.

The energy balance of an arbitrary part Π of the shell midsurface M can be described in analogy to the 3D energy balance, see [51], by the resultant surface fields [34] as

$$\frac{d}{dt}E = A + Q, \tag{3.1}$$

where E is the resultant total energy, A is the resultant mechanical power, and Q is the resultant heat supply. In (3.1), $E = \iint\limits_{\Pi} \rho\varepsilon \, da$, ρ is the undeformed surface mass density, ε is the internal surface energy density per unit undeformed surface mass, and A in the quasistatic process can be given by the relation

$$A = \iint\limits_{\Pi} (\boldsymbol{f} \cdot \boldsymbol{v} + \boldsymbol{c} \cdot \boldsymbol{\omega}) \, da + \int\limits_{\partial\Pi} (\boldsymbol{n}_\nu \cdot \boldsymbol{v} + \boldsymbol{m}_\nu \cdot \boldsymbol{\omega}) \, ds.$$

In (3.1), Q is defined as

$$Q = \iint\limits_{\Pi} \rho(q^+ + q^- + q_\Pi) \, da - \int\limits_{\partial\Pi\setminus\partial M_h} q_\nu \, ds - \int\limits_{\partial\Pi\cap\partial M_h} q^* \, ds,$$

where $q^\pm$ are the heat influx densities through the upper $(+)$ and lower $(-)$ shell faces defined as in [36, 37, 55], q_Π is the internal surface heat supply density, while q_ν and q^* are the heat supplies through the internal boundary contour $\partial\Pi$ and the external boundary contour ∂M_h, respectively. The field q_ν is defined through the surface heat influx vector $\boldsymbol{q}$ according to $q_\nu = \boldsymbol{q} \cdot \boldsymbol{\nu}$. We shall omit q^* in the following discussion.

From (3.1) we obtain the local energy balance at any regular point of M,

$$\rho\frac{d\varepsilon}{dt} = \rho(q^+ + q^- + q_\Pi) - \mathrm{Div}_s \, \boldsymbol{q} + \boldsymbol{N} \bullet \boldsymbol{E}^\circ + \boldsymbol{M} \bullet \boldsymbol{K}^\circ, \tag{3.2}$$

where $(\ldots)^\circ$ is the co-rotational time derivative, see [10, 13, 41],

$$\boldsymbol{E}^\circ = \boldsymbol{Q}\frac{d}{dt}\left(\boldsymbol{Q}^T\boldsymbol{E}\right), \qquad \boldsymbol{K}^\circ = \boldsymbol{Q}\frac{d}{dt}\left(\boldsymbol{Q}^T\boldsymbol{K}\right),$$

and the scalar product of two tensors $(\boldsymbol{A}, \boldsymbol{B}) \in E \otimes T_x M$ is defined by $\boldsymbol{A} \bullet \boldsymbol{B} = \operatorname{tr}(\boldsymbol{A}^T \boldsymbol{B})$.

Introducing the referential shell stress and couple stress tensors as well as the referential shell strain measures by the relations

$$\mathbf{N} = \boldsymbol{Q}^T \boldsymbol{N}, \qquad \mathbf{M} = \boldsymbol{Q}^T \boldsymbol{M}, \qquad \mathbf{E} = \boldsymbol{Q}^T \boldsymbol{E}, \qquad \mathbf{K} = \boldsymbol{Q}^T \boldsymbol{K},$$

one can rewrite (3.2) as

$$\rho \frac{d\varepsilon}{dt} = \rho(q^+ + q^- + q_\Pi) - \operatorname{Div}_s \boldsymbol{q} + \mathbf{N} \bullet \frac{d\mathbf{E}}{dt} + \mathbf{M} \bullet \frac{d\mathbf{K}}{dt}. \tag{3.3}$$

From (3.1) we also obtain the local balance equation at C

$$V[\![\rho\varepsilon]\!] + [\![\boldsymbol{n}_\nu \cdot \boldsymbol{v}]\!] + [\![\boldsymbol{m}_\nu \cdot \boldsymbol{\omega}]\!] - [\![\boldsymbol{q} \cdot \boldsymbol{\nu}]\!] = 0. \tag{3.4}$$

The relation (3.4) is the special case of the general resultant continuity condition proposed in [34], Sec. 3.6.

4. Clausius–Duhem inequality

We take the second law of the surface thermodynamics in the simple form proposed by MURDOCH [36, 37]

$$\frac{d}{dt} \iint\limits_\Pi \rho\eta \, da \geq \iint\limits_\Pi \rho \left(\frac{q^+}{T^+_{\text{ext}}} + \frac{q^-}{T^-_{\text{ext}}} + \frac{q_\Pi}{T} \right) da - \int\limits_{\partial\Pi} \frac{q_\nu}{T} \, ds, \tag{4.1}$$

where η is the surface entropy density. In (4.1), by T^+_{ext} and T^-_{ext} we denote temperatures of the external media surrounding the shell from above and below, respectively.

The local form of (4.1) at any regular point of M is

$$\rho \frac{d\eta}{dt} \geq \rho \left(\frac{q^+}{T^+_{\text{ext}}} + \frac{q^-}{T^-_{\text{ext}}} + \frac{q_\Pi}{T} \right) - \operatorname{Div}_s \left(\frac{1}{T} \boldsymbol{q} \right). \tag{4.2}$$

Solving (3.3) for q_Π and eliminating it from (4.2), we obtain the reduced dissipation inequality for the shell

$$\begin{aligned} \rho \frac{d\psi}{dt} \leq -\rho\eta \frac{dT}{dt} + \mathbf{N} \bullet \frac{d\mathbf{E}}{dt} + \mathbf{M} \bullet \frac{d\mathbf{K}}{dt} + T \operatorname{Grad}_s \left(\frac{1}{T} \right) \cdot \boldsymbol{q} \\ + \rho q^+ \left(1 - \frac{T}{T^+_{\text{ext}}} \right) + \rho q^- \left(1 - \frac{T}{T^-_{\text{ext}}} \right), \end{aligned} \tag{4.3}$$

where $\psi = \varepsilon - T\eta$ is the surface free energy density. Equation (4.3) is analogous to the reduced dissipation inequality in 3D continuum mechanics [51], Eq. (XV.2-2), or [52], Eq. (1.19).

From (4.1) also follows the jump condition at C,

$$V[\![\rho\eta]\!] - \left[\!\left[\frac{1}{T}\boldsymbol{q}\cdot\boldsymbol{\nu}\right]\!\right] \equiv \delta^2 \geq 0. \tag{4.4}$$

The quantity δ^2 represents creation of entropy at the interface C. If $\delta = 0$ then the phase transition is called reversible. In such a case the balance equation at C reduces to

$$V[\![\rho\eta]\!] - \left[\!\left[\frac{1}{T}\boldsymbol{q}\cdot\boldsymbol{\nu}\right]\!\right] = 0.$$

More complete local forms of (4.3) and (4.4) are discussed in [34], Sec. 3.7.

5. Thermodynamic continuity condition

Let us discuss the relations (2.3), (2.7), (3.4) and (4.4) for jumps of various fields at C. We remind that these relations should be satisfied for arbitrary shells, also elastic, thermoelastic and thermoviscoelastic. This is so because these relations either represent continuity as (2.7), or balance equations of some fields at the singular curve C quasistatically moving on the base surface M. Additionally, we assume that the temperature field T is continuous in M, that is $[\![T]\!] = 0$ at C.

Eliminating $[\![\boldsymbol{q}\cdot\boldsymbol{\nu}]\!]$ from (3.4) and (4.4), we obtain

$$V[\![\rho\psi]\!] + T\delta^2 + [\![\boldsymbol{n}_\nu\cdot\boldsymbol{v}]\!] + [\![\boldsymbol{m}_\nu\cdot\boldsymbol{\omega}]\!] = 0.$$

Using the identities

$$[\![\boldsymbol{n}_\nu\cdot\boldsymbol{v}]\!] = \langle\boldsymbol{n}_\nu\rangle\cdot[\![\boldsymbol{v}]\!] + [\![\boldsymbol{n}_\nu]\!]\cdot\langle\boldsymbol{v}\rangle, \qquad [\![\boldsymbol{m}_\nu\cdot\boldsymbol{\omega}]\!] = \langle\boldsymbol{m}_\nu\rangle\cdot[\![\boldsymbol{\omega}]\!] + [\![\boldsymbol{m}_\nu]\!]\cdot\langle\boldsymbol{\omega}\rangle,$$

where $\langle\ldots\rangle = \frac{1}{2}\left[(\ldots)_A + (\ldots)_B\right]$ is the mean value at C, and taking into account the static balance equations (2.3), we obtain the relation

$$V[\![\rho\psi]\!] + \langle\boldsymbol{n}_\nu\rangle\cdot[\![\boldsymbol{v}]\!] + \langle\boldsymbol{m}_\nu\rangle\cdot[\![\boldsymbol{\omega}]\!] + T\delta^2 = 0 \qquad \text{at } C,$$

or

$$V[\![\rho\psi]\!] + \boldsymbol{\nu}\cdot\boldsymbol{N}^T[\![\boldsymbol{v}]\!] + \boldsymbol{\nu}\cdot\boldsymbol{M}^T[\![\boldsymbol{\omega}]\!] + T\delta^2 = 0 \qquad \text{at } C.$$

For the coherent phase interface from (2.7) it follows that

$$T\delta^2 = -V\left\{[\![\rho\psi]\!] - \boldsymbol{\nu}\cdot\boldsymbol{N}^T[\![\boldsymbol{F}\boldsymbol{\nu}]\!] - \boldsymbol{\nu}\cdot\boldsymbol{M}^T[\![\boldsymbol{K}\boldsymbol{\nu}]\!]\right\} \qquad \text{at } C. \tag{5.1}$$

Assuming the relation $\boldsymbol{\nu}\cdot\boldsymbol{M}^T = \boldsymbol{0}$ used in [12], for the phase interface incoherent in rotations the following relation holds at C:

$$T\delta^2 = -V\left\{[\![\rho\psi]\!] - \boldsymbol{\nu}\cdot\boldsymbol{N}^T[\![\boldsymbol{F}\boldsymbol{\nu}]\!]\right\}. \tag{5.2}$$

Equations (5.1) and (5.2) can be rewritten using the Eshelby tensors, introduced in [12] within the general nonlinear theory of elastic shells, in the form

$$T\delta^2 = -V\boldsymbol{\nu}\cdot[\![\boldsymbol{C}]\!]\boldsymbol{\nu}, \tag{5.3}$$

where $\boldsymbol{C} = \boldsymbol{C}_c \equiv \rho\psi\mathbf{A} - \boldsymbol{N}^T\boldsymbol{F} - \boldsymbol{M}^T\boldsymbol{K}$ for the coherent interface and $\boldsymbol{C} = \boldsymbol{C}_i \equiv \rho\psi\mathbf{A} - \boldsymbol{N}^T\boldsymbol{F}$ for the interface incoherent in rotations. Here $\boldsymbol{A} = \mathbf{1} - \boldsymbol{\eta}\otimes\boldsymbol{\eta},\ \boldsymbol{\eta}\cdot\boldsymbol{A}\boldsymbol{\eta} = 0$, while $\mathbf{1} \in E\otimes E$ and $\boldsymbol{A} \in T_xM\otimes T_xM$ are metric tensors of the 3D space and of the undeformed base surface, respectively.

Equation (5.3) follows from the general balance for jumps of the fields. Hence, this equation should also be satisfied for any other type of the singular curve C describing a slow (quasistatic) motion of various defects such as, for example, shear bands, dislocations, etc. From the point of view of configurational mechanics [22, 35, 25], the quantity $\boldsymbol{\nu}\cdot[\![\boldsymbol{C}]\!]\boldsymbol{\nu}$ represents the configurational (or driving) force acting on C and responsible for its motion. In the papers [12, 27] we formulated the balance equations along the phase interface C in the equilibrium state. In [12] it was found that in equilibrium the condition $\boldsymbol{\nu}\cdot[\![\boldsymbol{C}]\!]\boldsymbol{\nu} = 0$ should be satisfied. This condition was necessary to find the position of the phase interface C in the equilibrium state.

According to the second principle of thermodynamics, the entropy production δ^2 remains always non-negative for all thermodynamic processes. Equation (5.3) allows us to postulate the kinetic equation, describing motion of the phase interface for all quasistatic processes, in the form

$$V = -\mathfrak{F}\left(\boldsymbol{\nu}\cdot[\![\boldsymbol{C}]\!]\boldsymbol{\nu}\right), \tag{5.4}$$

where $\mathfrak{F}$ is the non-negative definite kinetic function depending on the jump of $\boldsymbol{C}$ at C, i.e. $\mathfrak{F}(\varsigma) \geq 0$ for $\varsigma > 0$, where $\boldsymbol{C} = \boldsymbol{C}_c$ for the coherent interface and $\boldsymbol{C} = \boldsymbol{C}_i$ for the incoherent one. In the 3D theory of elasticity the kinetic equations of the type (5.4) were discussed in a number of papers among which let us note [1–4, 6, 15, 26, 29, 30, 33, 44]. Equation (5.4) can also be viewed as a kind of constitutive relation consistent with the thermodynamic requirement $\delta^2 \geq 0$.

One can also discuss other forms of the kinetic equation different from (5.4). As an example of such relations one can postulate a simpler relation $\boldsymbol{\nu}\cdot[\![\boldsymbol{C}]\!]\boldsymbol{\nu} = 0$ not only in equilibrium but also in quasistatic motion. From the physical point of view this would mean that the velocity towards phase equilibrium is assumed to be much higher than the characteristic velocity of deformation. In other words, the phase equilibrium at each quasistatic deformation is being attained much faster than the body deforms. One can also formulate more complex relations depending on the linear energy density or on other factors. In what follows we restrict our discussion to the kinetic equation (5.4).

Since in the equilibrium state $\boldsymbol{\nu}\cdot[\![\boldsymbol{C}]\!]\boldsymbol{\nu}=0$ and $V=0$, it is natural to require that $\mathfrak{F}(0)=0$. The form of the function $\mathfrak{F}$ depends mainly upon the type of phase transition, material behaviour, temperature and other factors. After [2], let us assume $\mathfrak{F}(\varsigma)$ in the form

$$\mathfrak{F}(\varsigma)=\begin{cases} k(\varsigma-\varsigma_0) & \varsigma\geq\varsigma_0,\\ 0 & -\varsigma_0<\varsigma<\varsigma_0,\\ k(\varsigma+\varsigma_0) & \varsigma\leq-\varsigma_0. \end{cases} \tag{5.5}$$

Here ς_0 describes the effects associated with nucleation of the new phase and action of the surface tension, see [2], and k is a positive kinetic factor. If $\varsigma_0=0$ then the function (5.5) reduces to the linear kinetic function $\mathfrak{F}(\varsigma)=k\varsigma$. Such a kinetic function was used for example in [54] in the stability analysis of two-phase continua.

Summarising, in the case of finite deformations the BVP for the shell undergoing phase transitions consists of the equilibrium equations and static boundary conditions (2.2) supplemented by appropriate kinematic boundary conditions for $\boldsymbol{u}$ and $\boldsymbol{Q}$, the energy transfer equation (3.2) with appropriate boundary conditions for T, as well as the balance equations (2.3), (2.7), (3.4) and (5.4) along the interface C. Equation (5.4) distinguishes the considered problem from the one for a compound shell, because in our case it is used to find position of the curvilinear interface C in its quasistatic motion. It is apparent that it is not possible to solve such a BVP without the constitutive equations expressing $\boldsymbol{N}$, $\boldsymbol{M}$ (or $\mathbf{N}$, $\mathbf{M}$), ψ, η, $\boldsymbol{q}$, $q^{\pm}$ in terms of deformation and temperature.

6. Thermoelastic shells

Let us discuss the simple case of thermoelastic shell. In this case $\mathbf{N}$, $\mathbf{M}$, ψ, η, $\boldsymbol{q}$, $q^{\pm}$ do not depend on prehistories of deformations and temperature but only upon their actual values

$$\begin{aligned} \mathbf{N}&=\mathbf{N}(\mathbf{E},\mathbf{K},T,\boldsymbol{g}), & \mathbf{M}&=\mathbf{M}(\mathbf{E},\mathbf{K},T,\boldsymbol{g}),\\ \psi&=\psi(\mathbf{E},\mathbf{K},T,\boldsymbol{g}), & \eta&=\eta(\mathbf{E},\mathbf{K},T,\boldsymbol{g}),\\ \boldsymbol{q}&=\boldsymbol{q}(\mathbf{E},\mathbf{K},T,\boldsymbol{g}), & q^{\pm}&=q^{\pm}(\mathbf{E},\mathbf{K},T,\boldsymbol{g}), \end{aligned} \tag{6.1}$$

where $\boldsymbol{g}=\operatorname{Grad}_s T$. One can check that the constitutive equations (6.1) satisfy the principle of material frame-indifference [51]. Notice that Eqs. (6.1) can also depend on $T^{\pm}_{\rm ext}$ as parameters.

Substituting (6.1) into (4.3), we obtain the inequality

$$\begin{aligned} \left(\rho\frac{\partial\psi}{\partial\mathbf{E}}-\mathbf{N}\right)\bullet\frac{d\mathbf{E}}{dt}+\left(\rho\frac{\partial\psi}{\partial\mathbf{K}}-\mathbf{M}\right)\bullet\frac{d\mathbf{K}}{dt}+\rho\left(\frac{\partial\psi}{\partial T}+\eta\right)\frac{dT}{dt}+\rho\frac{d\psi}{d\boldsymbol{g}}\cdot\frac{d\boldsymbol{g}}{dt}\\ \leq T\operatorname{Grad}_s\frac{1}{T}\cdot\boldsymbol{q}+\rho q^{+}\left(1-\frac{T}{T^{+}_{\rm ext}}\right)+\rho q^{-}\left(1-\frac{T}{T^{-}_{\rm ext}}\right). \end{aligned} \tag{6.2}$$

Applying the method of [51], from (6.2) follow the reduced constitutive equations of thermoelastic shells

$$\psi = \psi(\mathbf{E}, \mathbf{K}, T), \qquad \eta = -\frac{\partial \psi}{\partial T}, \qquad \mathbf{N} = \rho \frac{\partial \psi}{\partial \mathbf{E}}, \qquad \mathbf{M} = \rho \frac{\partial \psi}{\partial \mathbf{K}}, \tag{6.3}$$

and the inequality (6.2) restricts only the heat influxes

$$T \operatorname{Grad}_s \frac{1}{T} \cdot \boldsymbol{q} + \rho q^+ \left(1 - \frac{T}{T^+_{\text{ext}}}\right) + \rho q^- \left(1 - \frac{T}{T^-_{\text{ext}}}\right) \geq 0. \tag{6.4}$$

Inequality (6.4) is the 2D analogue in shell theory of the 3D Fourier inequality [51]. The inequality is satisfied for example by the following constitutive equations for $\boldsymbol{q}$ and $q^{\pm}$:

$$\boldsymbol{q} = -c \operatorname{Grad}_s T, \qquad q^{\pm} = -c^{\pm}(T - T^{\pm}_{\text{ext}}), \tag{6.5}$$

where c is the positive coefficient of heat conductivity of the shell in tangential direction, and $c^{\pm}$ are coefficients describing the heat exchange between the shell and the surrounding media according to the Newton law (heat influx through the surface is proportional to temperature difference).

For thermoelastic shells the local energy balance (3.3) takes the form of the heat conductivity balance

$$\rho T \frac{d\eta}{dt} = -\operatorname{Div}_s \boldsymbol{q} + \rho(q_\Pi + q^+ + q^-). \tag{6.6}$$

As an example of the constitutive description for the isotropic thermoelastic shell one may consider the following form of the surface free energy density:

$$\begin{aligned} 2\rho\psi = {} & \alpha_1 \operatorname{tr}^2 \mathbf{E}_\| + \alpha_2 \operatorname{tr} \mathbf{E}_\|^2 + \alpha_3 \operatorname{tr}\left(\mathbf{E}_\|^T \mathbf{E}_\|\right) + \alpha_4 \boldsymbol{\eta} \cdot \mathbf{E}\mathbf{E}^T \boldsymbol{\eta} + 2\alpha(T) \operatorname{tr} \mathbf{E}_\| \\ & + \beta_1 \operatorname{tr}^2 \mathbf{K}_\| + \beta_2 \operatorname{tr} \mathbf{K}_\|^2 + \beta_3 \operatorname{tr}\left(\mathbf{K}_\|^T \mathbf{K}_\|\right) + \beta_4 \boldsymbol{\eta} \cdot \mathbf{K}\mathbf{K}^T \boldsymbol{\eta} \\ & + 2\beta(T) \operatorname{tr} \mathbf{K}_\| + 2\rho\psi_0(T), \end{aligned} \tag{6.7}$$

$$\mathbf{E}_\| = \mathbf{E} - \mathbf{E}^T \boldsymbol{\eta}, \quad \mathbf{K}_\| = \mathbf{K} - \mathbf{K}^T \boldsymbol{\eta},$$

where $(\ldots)_\|$ means the part of the tensor $(\ldots)$ in the tangent space $T_x M \otimes T_x M$ to M at $x \in M$. This equation is a direct generalization of the density of the physically linear elastic shell [13]. In (6.7) the function $\psi_0(T)$ describes dependence of the free energy density on temperature in the undeformed state, while $\alpha(T)$ and $\beta(T)$ are connected with the shell expansion due to temperature changes. In the simplest case these functions are linear in T. In the expression (6.7) we also have eight constitutive factors α_k, β_k $(k = 1, 2, 3, 4)$ which can depend on T as well, in general.

The function (6.7) generates the following constitutive equations for the isotropic thermoelastic shell:

$$
\begin{aligned}
&\mathbf{N} = \alpha_1 \boldsymbol{A} \operatorname{tr} \mathbf{E}_\parallel + \alpha_2 \mathbf{E}_\parallel^T + \alpha_3 \mathbf{E}_\parallel + \alpha_4 \boldsymbol{\eta} \otimes \mathbf{E}^T \boldsymbol{\eta} + \alpha(T) \boldsymbol{A}, \\
&\mathbf{M} = \beta_1 \boldsymbol{A} \operatorname{tr} \mathbf{K}_\parallel + \beta_2 \mathbf{K}_\parallel^T + \beta_3 \mathbf{K}_\parallel + \beta_4 \boldsymbol{\eta} \otimes \mathbf{K}^T \boldsymbol{\eta} + \beta(T) \boldsymbol{A}, \\
&\rho\eta = -\frac{d\alpha}{dT} \operatorname{tr} \mathbf{E}_\parallel - \frac{d\beta}{dT} \operatorname{tr} \mathbf{K}_\parallel - \rho \frac{d\psi_0}{dT}.
\end{aligned} \tag{6.8}
$$

In [10] the following relations for the elastic moduli appearing in (6.7) and (6.8) were used:

$$
\begin{aligned}
&\alpha_1 = C\nu, \quad \alpha_2 = 0, \quad \alpha_3 = C(1-\nu), \quad \alpha_4 = \alpha_s C(1-\nu), \\
&\beta_1 = D\nu, \quad \beta_2 = 0, \quad \beta_3 = D(1-\nu), \quad \beta_4 = \alpha_t D(1-\nu), \\
&C = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)},
\end{aligned} \tag{6.9}
$$

where E and ν are the Young modulus and the Poisson ratio of the bulk material, respectively, α_s and α_t are dimensionless coefficients, while h is the shell thickness.

The constitutive equations (6.8) are written assuming that $\psi = 0$ in the absence of deformation. In some cases of phase transition it is necessary to write the constitutive equations for deformation relative to some intermediate configuration following from a certain value of the phase transformation strain, see for example [19, 54]. In the case of small deformations discussed here, such constitutive equations follow from (6.8) if we change there $\mathbf{E}$ and $\mathbf{K}$ for $\mathbf{E} - \mathbf{E}_p$ and $\mathbf{K} - \mathbf{K}_p$, respectively, where $\mathbf{E}_p$ and $\mathbf{K}_p$ are now values of the surface strain measures corresponding to shell deformation during the phase transition without loading.

7. Viscoelastic shells of differential type

As an example of shells with memory effects let us discuss the constitutive equations ([34], Chapter 9)

$$
\begin{aligned}
&\mathbf{N} = \mathbf{N}\left(\mathbf{E}, \mathbf{K}, \frac{d\mathbf{E}}{dt}, \frac{d\mathbf{K}}{dt}, T, \boldsymbol{g}\right), \qquad \mathbf{M} = \mathbf{M}\left(\mathbf{E}, \mathbf{K}, \frac{d\mathbf{E}}{dt}, \frac{d\mathbf{K}}{dt}, T, \boldsymbol{g}\right), \\
&\psi = \psi\left(\mathbf{E}, \mathbf{K}, \frac{d\mathbf{E}}{dt}, \frac{d\mathbf{K}}{dt}, T, \boldsymbol{g}\right), \qquad \eta = \eta\left(\mathbf{E}, \mathbf{K}, \frac{d\mathbf{E}}{dt}, \frac{d\mathbf{K}}{dt}, T, \boldsymbol{g}\right), \\
&\boldsymbol{q} = \boldsymbol{q}\left(\mathbf{E}, \mathbf{K}, \frac{d\mathbf{E}}{dt}, \frac{d\mathbf{K}}{dt}, T, \boldsymbol{g}\right), \qquad q^{\pm} = q^{\pm}\left(\mathbf{E}, \mathbf{K}, \frac{d\mathbf{E}}{dt}, \frac{d\mathbf{K}}{dt}, T, \boldsymbol{g}\right).
\end{aligned} \tag{7.1}
$$

These constitutive equations satisfy the principle of material frame-indifference [51] as well. They are counterparts in the general theory of shells of the Voigt viscoelastic material model. Substituting (7.1) into (4.3), we obtain the inequality

$$\left(\rho\frac{\partial\psi}{\partial\mathbf{E}}-\mathbf{N}\right)\bullet\frac{d\mathbf{E}}{dt}+\left(\rho\frac{\partial\psi}{\partial\mathbf{K}}-\mathbf{M}\right)\bullet\frac{d\mathbf{K}}{dt}+\rho\left(\frac{\partial\psi}{\partial T}+\eta\right)\frac{dT}{dt}+\rho\frac{d\psi}{d\boldsymbol{g}}\cdot\frac{d\boldsymbol{g}}{dt}$$
$$+\rho\frac{\partial\psi}{\partial(d\mathbf{E}/dt)}\bullet\frac{d^2\mathbf{E}}{dt^2}+\rho\frac{\partial\psi}{\partial(d\mathbf{K}/dt)}\bullet\frac{d^2\mathbf{K}}{dt^2}$$
$$\leq T\,\mathrm{Grad}_s\,\frac{1}{T}\cdot\boldsymbol{q}+\rho q^+\left(1-\frac{T}{T^+_{\mathrm{ext}}}\right)+\rho q^-\left(1-\frac{T}{T^-_{\mathrm{ext}}}\right). \tag{7.2}$$

Let us represent the constitutive equations for $\mathbf{N}$ and $\mathbf{M}$ as the sum of equilibrium and dissipative parts

$$\mathbf{N}=\mathbf{N}_E+\mathbf{N}_D,\quad \mathbf{M}=\mathbf{M}_E+\mathbf{M}_D, \tag{7.3}$$

where

$$\mathbf{N}_E=\mathbf{N}_E(\mathbf{E},\mathbf{K},T,\boldsymbol{g}),\qquad \mathbf{M}_E=\mathbf{M}_E(\mathbf{E},\mathbf{K},T,\boldsymbol{g}),$$
$$\mathbf{N}_D=\mathbf{N}_D\left(\mathbf{E},\mathbf{K},\frac{d\mathbf{E}}{dt},\frac{d\mathbf{K}}{dt},T,\boldsymbol{g}\right),\qquad \mathbf{M}_D=\mathbf{M}_D\left(\mathbf{E},\mathbf{K},\frac{d\mathbf{E}}{dt},\frac{d\mathbf{K}}{dt},T,\boldsymbol{g}\right),$$

with restrictions

$$\mathbf{N}_D\left(\mathbf{E},\mathbf{K},\mathbf{0},\mathbf{0},T,\boldsymbol{g}\right)=\mathbf{0},\qquad \mathbf{M}_D\left(\mathbf{E},\mathbf{K},\mathbf{0},\mathbf{0},T,\boldsymbol{g}\right)=\mathbf{0}.$$

Substituting (7.3) into the inequality (7.2) we obtain

$$\left(\rho\frac{\partial\psi}{\partial\mathbf{E}}-\mathbf{N}_E\right)\bullet\frac{d\mathbf{E}}{dt}+\left(\rho\frac{\partial\psi}{\partial\mathbf{K}}-\mathbf{M}_E\right)\bullet\frac{d\mathbf{K}}{dt}+\rho\left(\frac{\partial\psi}{\partial T}+\eta\right)\frac{dT}{dt}$$
$$+\rho\frac{d\psi}{d\boldsymbol{g}}\cdot\frac{d\boldsymbol{g}}{dt}+\rho\frac{\partial\psi}{\partial(d\mathbf{E}/dt)}\bullet\frac{d^2\mathbf{E}}{dt^2}+\rho\frac{\partial\psi}{\partial(d\mathbf{K}/dt)}\bullet\frac{d^2\mathbf{K}}{dt^2}-\mathbf{N}_D\bullet\frac{d\mathbf{E}}{dt}-\mathbf{M}_D\bullet\frac{d\mathbf{K}}{dt}$$
$$\leq T\,\mathrm{Grad}_s\,\frac{1}{T}\cdot\boldsymbol{q}+\rho q^+\left(1-\frac{T}{T^+_{\mathrm{ext}}}\right)+\rho q^-\left(1-\frac{T}{T^-_{\mathrm{ext}}}\right). \tag{7.4}$$

From (7.4) it follows that

$$\psi=\psi(\mathbf{E},\mathbf{K},T),\qquad \eta=-\frac{\partial\psi}{\partial T},\qquad \mathbf{N}_E=\rho\frac{\partial\psi}{\partial\mathbf{E}},\qquad \mathbf{M}_E=\rho\frac{\partial\psi}{\partial\mathbf{K}}, \tag{7.5}$$

and the dissipation inequality (7.4) reduces to

$$(7.6)\quad \mathbf{N}_D \bullet \frac{d\mathbf{E}}{dt} + \mathbf{M}_D \bullet \frac{d\mathbf{K}}{dt} + T \operatorname{Grad}_s \frac{1}{T} \cdot \boldsymbol{q} + \rho q^+ \left(1 - \frac{T}{T^+_{\text{ext}}}\right) + \rho q^- \left(1 - \frac{T}{T^-_{\text{ext}}}\right) \geq 0.$$

Indeed, since the second derivatives $d^2\mathbf{E}/dt^2$ and $d^2\mathbf{K}/dt^2$ can take arbitrary values, and signs and their multipliers do not depend on these derivatives, in order to satisfy (7.4) it is necessary that

$$\frac{\partial \psi}{\partial (d\mathbf{E}/dt)} = \mathbf{0}, \quad \frac{\partial \psi}{\partial (d\mathbf{K}/dt)} = \mathbf{0}.$$

Otherwise, by choosing appropriate signs and values of the second derivatives we can give the inequality any of the two signs. The proof idea is thus reduced to the statement that the inequality $0 \leq ax + b$ is satisfied for any x if and only if $a = 0$, $b \geq 0$. Analogously we can show that $d\psi/d\boldsymbol{g} = \mathbf{0}$. Hence, $\psi = \psi(\mathbf{E}, \mathbf{K}, T)$. Repeating the discussion concerning the first time derivatives of $\mathbf{E}$ and $\mathbf{K}$, we obtain that their multipliers should vanish as well. As a result, from (7.4) follow (7.5) and (7.6) indeed.

For thermoviscoelastic shells the local energy balance (3.3) becomes

$$(7.7)\qquad \rho T \frac{d\eta}{dt} = -\operatorname{Div}_s \boldsymbol{q} + \mathbf{N}_D \bullet \frac{d\mathbf{E}}{dt} + \mathbf{M}_D \bullet \frac{d\mathbf{K}}{dt} + \rho(q_\Pi + q^+ + q^-).$$

As an example of simple constitutive relations of the type (7.3), one may consider the constitutive equations of thermoviscoelastic isotropic shells consisting of the relations (6.8) and (6.5) for the equilibrium parts and the linear constitutive equations for dissipative parts

$$\mathbf{N}_D = \zeta_1 \boldsymbol{A} \operatorname{tr} \frac{d\mathbf{E}_\parallel}{dt} + \zeta_2 \frac{d\mathbf{E}_\parallel^T}{dt} + \zeta_3 \frac{d\mathbf{E}_\parallel}{dt} + \zeta_4 \boldsymbol{\eta} \otimes \frac{d\mathbf{E}^T}{dt} \boldsymbol{\eta},$$

$$\mathbf{M}_D = \mu_1 \boldsymbol{A} \operatorname{tr} \frac{d\mathbf{K}_\parallel}{dt} + \mu_2 \frac{d\mathbf{K}_\parallel^T}{dt} + \mu_3 \frac{d\mathbf{K}_\parallel}{dt} + \mu_4 \boldsymbol{\eta} \otimes \frac{d\mathbf{K}^T}{dt} \boldsymbol{\eta},$$

where ζ_k and μ_k $(k = 1, 2, 3, 4)$ are the viscoelastic coefficients. If one takes the constitutive equations (6.5), it is possible to show that the inequality (7.6) is equivalent to the following inequalities:

$$2\zeta_1 + \zeta_2 + \zeta_3 \geq 0, \qquad \zeta_2 + \zeta_3 \geq 0, \qquad \zeta_3 - \zeta_2 \geq 0, \qquad \zeta_4 \geq 0,$$
$$2\mu_1 + \mu_2 + \mu_3 \geq 0, \qquad \mu_2 + \mu_3 \geq 0, \qquad \mu_3 - \mu_2 \geq 0, \qquad \mu_4 \geq 0.$$

These moduli can also be taken in the form analogous to (6.9) if one exchanges E and ν there for the corresponding viscoelastic moduli of the bulk material.

8. Example: Tension of two-phase tube

Let us discuss the thin circular cylindrical shell of length L and radius R made of an elastic material undergoing phase transition. The shell is extended by forces P uniformly distributed at the right shell boundary, Fig. 3. The left shell boundary at $z = 0$ is clamped.

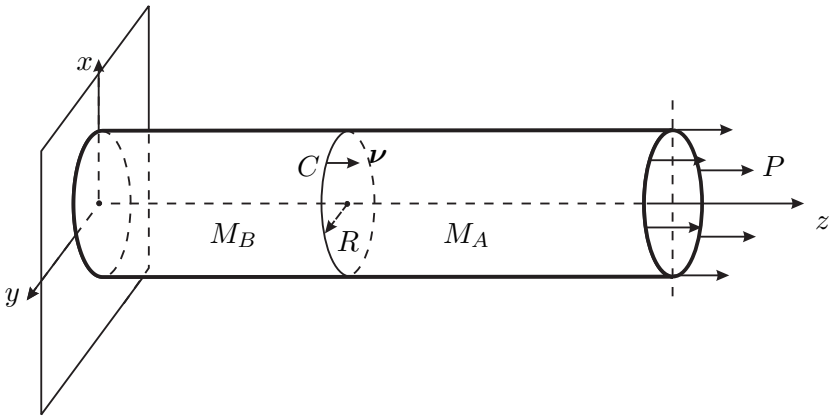

FIG. 3. Tension of the two-phase thin-walled tube.

We assume that the shell deformation is infinitesimal so that for the strain measures we can use the relations $(2.4)_{1,2}$ with (2.5). In such a case we approximately have $\boldsymbol{N} \cong \mathbf{N}$, $\boldsymbol{M} \cong \mathbf{M}$, $\boldsymbol{E} \cong \mathbf{E}$, $\boldsymbol{K} \cong \mathbf{K}$.

At $z = 0$ the kinematic boundary conditions

$$\boldsymbol{u} = \mathbf{0}, \qquad \boldsymbol{\varphi} = \mathbf{0} \tag{8.1}$$

are satisfied, and at $z = L$ we use the static boundary conditions

$$\boldsymbol{N}\boldsymbol{\nu} = P\boldsymbol{\nu}, \qquad \boldsymbol{M}\boldsymbol{\nu} = \mathbf{0}, \qquad (\boldsymbol{\nu} = \mathbf{i}_3). \tag{8.2}$$

Under such loading there is an axisymmetric deformation state

$$\boldsymbol{u} = u(z)\mathbf{e}_z + w(z)\mathbf{e}_r, \qquad \boldsymbol{\varphi} = \varphi(z)\mathbf{e}_\phi, \tag{8.3}$$

where $\mathbf{e}_r = \cos\phi\mathbf{i}_1 + \sin\phi\mathbf{i}_2$, $\mathbf{e}_\phi = -\sin\phi\mathbf{i}_1 + \cos\phi\mathbf{i}_2$, $\mathbf{e}_z = \mathbf{i}_3$ are the unit base vectors of cylindrical system of coordinates. With (8.3) the linearized strain measures are

$$\begin{aligned} \mathbf{E} &= \operatorname{Grad}_s \boldsymbol{u} - \boldsymbol{\varphi} \times \boldsymbol{A} = u'\mathbf{e}_z \otimes \mathbf{e}_z + (w' - \varphi)\mathbf{e}_r \otimes \mathbf{e}_z + \frac{w}{R}\mathbf{e}_\phi \otimes \mathbf{e}_\phi, \\ \mathbf{K} &\equiv \operatorname{Grad}_s \boldsymbol{\varphi} = \varphi'\mathbf{e}_\phi \otimes \mathbf{e}_z - \frac{\varphi}{R}\mathbf{e}_r \otimes \mathbf{e}_\phi, \end{aligned} \tag{8.4}$$

where $(\ldots)' = \dfrac{\partial}{\partial z}(\ldots)$.

The surface stress measures $\mathbf{N}$ and $\mathbf{M}$ of the axisymmetric stress state can be represented by

$$\begin{aligned}\mathbf{N} &= N_{zz}\mathbf{e}_z\otimes\mathbf{e}_z + N_{\phi\phi}\mathbf{e}_\phi\otimes\mathbf{e}_\phi + N_{rz}\mathbf{e}_r\otimes\mathbf{e}_z,\\ \mathbf{M} &= M_{\phi z}\mathbf{e}_\phi\otimes\mathbf{e}_z + M_{z\phi}\mathbf{e}_z\otimes\mathbf{e}_\phi + M_{r\phi}\mathbf{e}_r\otimes\mathbf{e}_z.\end{aligned} \tag{8.5}$$

The equilibrium conditions (2.2) with $\boldsymbol{f}=\boldsymbol{c}=\mathbf{0}$ take the simple form

$$\begin{gathered}N'_{zz}=0,\qquad N'_{rz}-\frac{N_{\phi\phi}}{R}=0,\qquad M'_{\phi z}+\frac{M_{r\phi}}{R}+N_{rz}=0,\qquad \text{in } M\backslash C,\\ N_{zz}=P,\qquad N_{rz}=0,\qquad M_{\phi z}=0\qquad \text{at } z=L.\end{gathered} \tag{8.6}$$

From $(8.6)_1$ it immediately follows that $N_{zz}=P$, $\forall z\in[0,L]$.

Let us assume that the deformation process is isothermic, and additionally that $T=T^+_{\rm ext}=T^-_{\rm ext}=\text{const.}$ and $q_\Pi=0$. In such a case the energy balance Eq. (6.6) or (7.7) is not discussed and the problem is reduced to the stress-induced phase transitions.

Let us consider the following free energy densities for the elastic phases A, B:

$$\begin{aligned}2\rho\psi^{A,B} &= \alpha_1^{A,B}\mathrm{tr}^{\,2}\tilde{\mathbf{E}}_\parallel + \alpha_2^{A,B}\mathrm{tr}\,\tilde{\mathbf{E}}_\parallel^2 + \alpha_3^{A,B}\mathrm{tr}\left(\tilde{\mathbf{E}}_\parallel^T\tilde{\mathbf{E}}_\parallel\right) + \alpha_4^{A,B}\boldsymbol{\eta}\cdot\mathbf{E}\mathbf{E}^T\boldsymbol{\eta}\\ &+ \beta_1^{A,B}\mathrm{tr}^{\,2}\mathbf{K}_\parallel + \beta_2^{A,B}\mathrm{tr}\,\mathbf{K}_\parallel^2 + \beta_3^{A,B}\mathrm{tr}\left(\mathbf{K}_\parallel^T\mathbf{K}_\parallel\right) + \beta_4^{A,B}\boldsymbol{\eta}\cdot\mathbf{K}\mathbf{K}^T\boldsymbol{\eta} + 2\rho\psi_0^{A,B}.\end{aligned} \tag{8.7}$$

Here $\tilde{\mathbf{E}}=\mathbf{E}-\mathbf{E}_p^{A,B}$, where $\mathbf{E}_p^{A,B}$ is a phase transformation strain. We assume that $\mathbf{E}_p^{A,B}=\epsilon_p^{A,B}\mathbf{A}$ with $\epsilon_p^B=0$. Such a phase transformation strain corresponds to an isotropic extension of a material under the phase transition. The relations (8.7) mean that both material phases differ by values of the elastic moduli and free energy densities in the undeformed state.

According to the assumption of axisymmetric deformation, we search for the phase interface C in the form of the circle with radius R, the position of which on M is given by the equation $z=\ell(t)$, $0\le\ell(t)\le L$. Hence, $V=\dot{\ell}(t)$. For the coherent interface, the kinematic and static balance equations on C, together with the kinetic equation, lead to the relations

$$\begin{gathered}[\![u]\!]=[\![w]\!]=0,\quad [\![\varphi]\!]=0,\quad [\![N_{zz}]\!]=0,\quad [\![N_{rz}]\!]=0,\quad [\![M_{\phi z}]\!]=0,\\ \dot{\ell}=-\mathfrak{F}(\varsigma),\quad \varsigma=[\![\rho\psi-N_{zz}(1+u')-N_{rz}w'-M_{\phi z}\varphi']\!]\quad \text{at } z=\ell(t).\end{gathered} \tag{8.8}$$

For the phase interface incoherent in rotations we obtain

$$\begin{gathered}[\![u]\!]=[\![w]\!]=0,\quad [\![N_{zz}]\!]=0,\quad [\![N_{rz}]\!]=0,\quad M_{\phi z}=0,\\ \dot{\ell}=-\mathfrak{F}(\varsigma),\quad \varsigma=[\![\rho\psi-N_{zz}(1+u')-N_{rz}w']\!]\quad \text{at } z=\ell(t).\end{gathered} \tag{8.9}$$

The ordinary differential equation $\dot{\ell} = -\mathfrak{F}(\varsigma)$ should be supplemented by the initial condition $\ell(0) = \ell_0$. The parameter ℓ_0 indicates the place within the cylinder in which the phase transition occurs first. In this example we assume that ℓ_0 describes the cylinder edges.

As a result, the discussed example is reduced to solving the boundary-value problem consisting of the system of equilibrium equations $(8.6)_1$, the constitutive equations (8.7), the boundary conditions (8.1) and (8.2) as well as the compatibility conditions (8.8) or (8.9) along C.

Let us discuss pure tension of the elastic one-phase cylinder. With notation (6.9), the equilibrium equations and the constitutive equations can be written as the following system of ordinary differential equations:

$$
\begin{aligned}
u' &= \frac{P}{C} - \nu\frac{w}{R} + \epsilon_p, \\
N'_{rz} &= \frac{N_{\phi\phi}}{R}, \qquad & N_{\phi\phi} &= \nu P + C(1-\nu^2)\frac{w}{R}, \\
M'_{\phi z} &= -\frac{M_{r\phi}}{R} - N_{rz}, \qquad & M_{r\phi} &= -\beta_4\frac{\varphi}{R}, \\
w' &= \frac{N_{rz}}{\alpha_4} + \varphi, \qquad & \varphi' &= \frac{M_{\phi z}}{D(1-\nu)}.
\end{aligned}
\tag{8.10}
$$

This is the system of five ODE with constant coefficients, expressed in terms of independent functions u, w, φ, N_{rz}, $M_{\phi z}$. The solution of $(8.10)_1$ can be found by direct integration when the other four equations have been solved.

The system (8.10) has always the particular solution

$$
u(z) = \left(\frac{P}{Eh} + \epsilon_p\right) z + \text{const}, \qquad w = -\nu u' R \equiv -\left(\frac{P}{Eh} + \epsilon_p\right)\nu R, \tag{8.11}
$$

for which $N_{rz} = 0$, $N_{\phi\phi} = 0$, $\varphi = 0$, $\mathbf{M} = \mathbf{0}$. This solution describes the axisymmetric membrane equilibrium state of the cylinder. In the two-phase cylinder such a solution is possible only when $\nu_A = \nu_B = 0$ or $\nu_A/E_A h_A = \nu_B/E_B h_B$ and $\epsilon_p^A = \epsilon_p^B$ since otherwise, according to (8.11), deformations of parts A and B would not coincide: $w_A \neq w_B$. This means that the pure membrane equilibrium state is not possible, in general, in the axisymmetric problem of the cylinder. Although (8.11) can adequately describe the solution far from the shell boundaries and far from the interface curve, for the full description of the two-phase cylinder one should construct the full solution of (8.10).

To illustrate the 2D model of phase transitions in the cylinder, in this paper we restrict ourselves to the simplest case when $\nu_A = \nu_B = 0$. Then the equilibrium solution of the two-phase shell is (8.11). Since in this case $\varphi = 0$, $\mathbf{M} = \mathbf{0}$, there is no difference between the coherent interface and the interface

incoherent in rotations, and this problem becomes entirely analogous to the 1D problem discussed by ABEYARATNE and KNOWLES [2] as a model problem of the 3D continuum model. If indices A and B denote solutions for different material phases, from (8.6) with the boundary conditions $u_B(0) = 0$, $u_B(\ell) = u_A(\ell)$ it follows that

$$u_B(z) = \frac{P}{C_B} z, \qquad u_A(Z) = \left(\frac{P}{C_A} + \epsilon_p^A\right) z + P\ell \left(\frac{1}{C_B} - \frac{1}{C_A}\right) - \epsilon_p^A \ell. \tag{8.12}$$

Then the free energy densities take the form

$$\begin{aligned} \rho\psi^B &= \frac{C_B}{2} E_B^2, & \rho\psi^A &= \frac{C_A}{2}(E_A - \epsilon_p^A)^2 + \Delta, \\ E_A &\equiv u_A' = \frac{P}{C_A} + \epsilon_p^A, & E_B &\equiv u_B' = \frac{P}{C_B}. \end{aligned} \tag{8.13}$$

Since the initial values of the free energies can be assumed to be arbitrary, in (8.13) we have assumed $\rho\psi_0^B = 0$, $\rho\psi_0^A = \Delta$. The graphical illustration of $\psi^{A,B}$ as functions of the stretch E is given in Fig. 4.

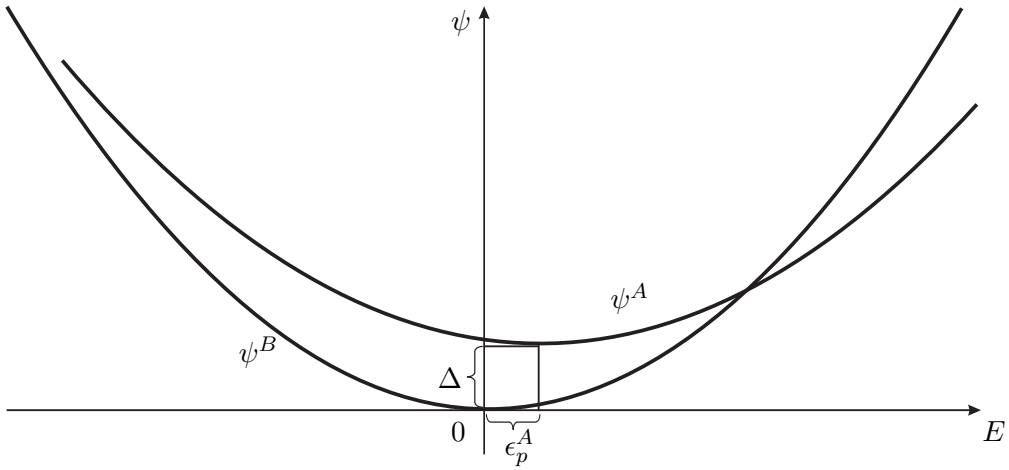

FIG. 4. Free energy densities.

For the discussed cylinder the variable ς becomes

$$\varsigma = [\![\rho\psi - N_{zz} u']\!] = [\![\rho\psi]\!] - P[\![u']\!] = \frac{P^2}{2}\left(\frac{1}{C_A} - \frac{1}{C_B}\right) + P\epsilon_p^A - \Delta. \tag{8.14}$$

The equilibrium position of the interface should follow from the equation $\varsigma = 0$. For the uniform stretch state it is easy to see that this equation is

satisfied by only one value of the force

$$P = P^* \equiv \frac{\sqrt{\epsilon_p^{A\,2} + 2\left(\frac{1}{C_A} - \frac{1}{C_B}\right)\Delta} - \epsilon_p^A}{\frac{1}{C_A} - \frac{1}{C_B}}. \tag{8.15}$$

Thus, in this case it is impossible to determine uniquely the position ℓ of the interface, because with $P = P^*$ the value of ℓ can be arbitrary within $[0, L]$. For the sake of simplicity we consider below only the case $\epsilon_p^A = 0$.

Let us consider the dependence of displacements of the right edge of the cylinder upon P. From $(8.12)_2$ we obtain that

$$P = C_A \left[1 + \frac{\ell}{L}\left(\frac{C_A}{C_B} - 1\right)\right]^{-1} E_L, \tag{8.16}$$

where $\ell \in [0, L]$, $E_L = u_A(L)/L$. The cases $\ell = 0$ and $\ell = L$ correspond to the cylinder consisting entirely of the phases A and B, respectively.

The relation how the force P depends on deformation in the equilibrium states follows from (8.16). It is illustrated in Fig. 5. This relation takes the form of piecewise-straight line $OABC$. The segment OA corresponds to deformation of the cylinder consisting of the phase B and is defined by $P = C_B E_L$. When P reaches the value P^*, it becomes possible to have different equilibrium states with two phases of the material. For definiteness, we assume that during loading the new phase A is first created at the right boundary of the cylinder, $z = L$. Then the two-phase cylinder behaves similarly as an elastic-perfectly plastic one: the deformation increases with constant value $P = P^*$ of the extension force, see the segment AB in Fig. 4. When P increases above P^*, the deformation of the cylinder follows the segment BC corresponding to the one-phase A state. The segment BC is more inclined than the one OA, since it is a part of the straight line $P = C_A E_L$ with $C_A < C_B$. For decreasing P through P^* up to 0, the equilibrium states follow the graph in the reverse order: $CBAO$.

Let us now discuss the quasistatic motion of C governed by the kinetic equation (5.4). The loading is assumed to be linearly increasing in time: $P = P_0 t$, $P_0 = \text{const}$, while the unloading rule to be again linear in time: $P = -P_0 t$. By changing variables $t = \pm P/P_0$ the kinetic equation can be written as

$$\pm P_0 \frac{d\ell}{dP} = -\mathfrak{F}(\varsigma). \tag{8.17}$$

In Fig. 5 we show the respective graphs AB' for the loading and BA' for the unloading obtained from (8.17). With the kinetic equation the phase transition process in the cylinder proceeds as follows. If $P \in (0, P^*)$, the cylinder deforms as consisting entirely of the phase B. This is the segment OA in Fig. 5.

When $P = P^*$, the new phase A is nucleated at the right boundary, and when the force P is growing above P^* the interface is moving towards the left boundary. This behaviour is described by the path AB' in Fig. 5. After reaching B' the cylinder consists entirely of the phase A and deforms further according to the segment $B'C$. In the process of unloading the cylinder deforms according to the path $CBA'O$.

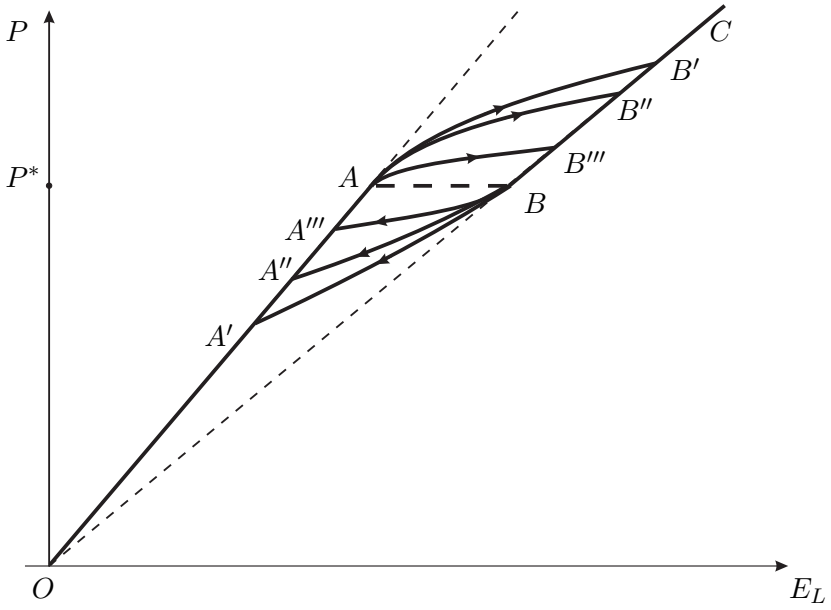

FIG. 5. $P - E$ curves for two-phase shell for different values of $\hat{k}$.

As a result, in the deformation process of the two-phase cylinder we observe the existence of the hysteresis loop $AB'BA'$ characteristic to the behaviour of phase transition of martensitic type. The size of the loop depends essentially on the form of function $\mathfrak{F}$, and particularly upon values of the kinetic factor k and the parameter P_0 determining the loading velocity. When $\hat{k} \equiv k/P_0$ increases, the area of hysteresis loop decreases. Examples of several deformation paths for different values of $\hat{k}$ are given in Fig. 5. It is seen that with the growing $\hat{k}$ we obtain the narrowing loops $AB'BA'$, $AB''BA''$, $AB'''BA'''$, etc. The limit $\hat{k} \to \infty$ corresponds both to the infinitely large kinetic factor $k \to \infty$ and to the infinitely small loading velocity $P_0 \to 0$. In the limit $\hat{k} \to \infty$ the hysteresis loop reduces to the equilibrium segment AB. This means in particular that with the infinitely small loading velocity the deformation follows the equilibrium path $OABC$.

We have discussed above the case $\varsigma_0 = 0$. Let us now analyse the cylinder deformation when $\varsigma_0 \neq 0$. Remember that ς_0 is associated with threshold

effects when the new phase is nucleated, [2]. For nucleation to occur it is necessary for the driving force ς to exceed some threshold value ς_0. It follows from (8.14) that $\varsigma = \pm\varsigma_0$ when $P = P^*_\pm$, and the inequality $P^*_- < P^* < P^*_+$ holds true. At the beginning the cylinder deforms similarly as above following the linear rule $P = C_B E_L$. In Fig. 6 this corresponds to the segment OAA_+. When $P = P^*_+$ the new phase A nucleates at the right boundary. The corresponding relation $P(E_L)$ is shown as the path A_+B_+ in Fig. 6. When ℓ attains the zero value, the cylinder becomes entirely consisting of the one phase A. For $P > P^*_+$ the cylinder deforms again according to the linear rule $P = C_A E_L$ (the segment B_+C).

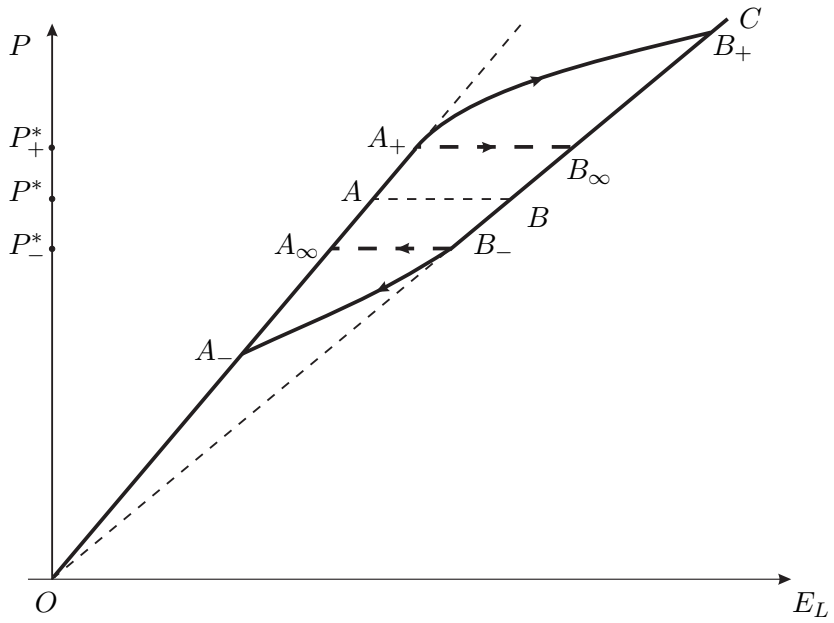

FIG. 6. $P - E$ curves for $\varsigma_0 \neq 0$.

In the unloading process $P = -P_0 t$ the cylinder, initially consisting entirely of the phase A, deforms according to the linear rule $P = C_A E_L$ until P attains the value P^*_-. Its further quasistatic motion is described by the kinetic equation with the initial condition $\ell(P^*_-) = 0$. The corresponding relation $P(E_L)$ is illustrated in Fig. 6 as the path B_-A_-. When ℓ reaches the value L, the cylinder becomes again consisting entirely of one phase B and its further deformation is governed by the linear rule $P = C_B E_L$ (the segment A_-O).

With $\varsigma_0 \neq 0$, the size of hysteresis loop becomes larger with the growing value of ς_0. But in this behaviour there is a qualitative difference as compared with the case $\varsigma_0 = 0$. When $\varsigma_0 \neq 0$, growing values of $\hat{k}$ also lead to decreasing areas

of the hysteresis loops. But for $\hat{k} \to \infty$ the limiting paths reduce to two different respective segments A_+B_∞ and B_-A_∞. In other words, the hysteresis loop takes place also in this limit case corresponding to the infinite kinetic factor and to the infinitely small loading velocity. Such limit cases describe the so-called rate-independent phase transitions observed in martensitic materials, see for example [2, 6, 28].

The simple example discussed here indicates that the general theory of shells with PT developed in this paper allows one to solve complex shell problems and obtain the results which in the special 1D case generalise the results reported in [1, 2, 6, 15, 26, 29] based on the assumed 1D models of PT in 3D elastic solids. In particular, the present example shows which assumptions about the stress state and the bulk material properties can lead to the 1D models discussed in those papers.

We have solved the problem of PT in the extended elastic circular cylinder using several simplifying assumptions about the external loads and the form of constitutive equations. In particular, we have assumed that $\nu_A = \nu_B = 0$. If this were not the case, we would obtain also additional solutions of the boundary layer type near the shell edges and near the phase interface. These solutions may considerably complicate the motion of the interface itself. Also a more complete loading applied on the shell edges (e.g. boundary shear, bending and/or torsion) might considerably complicate the solution. Additionally, in this example we have implicitly assumed that there is only one phase interface; it is possible to remove this assumption and analyse the problem with several moving phase boundaries as well. Detailed analysis of quasistatic deformation of the elastic circular cylindrical tube capable of undergoing phase transformations with the full account of the factors mentioned above is beyond the aim of this paper and should be presented separately.

9. Conclusions

We have worked out the complete 2D model of quasistatic deformations of the non-linear thermoelastic and thermoviscoelastic shells made of materials capable of undergoing phase transformations of martensitic type.

The new balance equations of linear momentum, angular momentum, and energy as well as the dissipation inequality at the curvilinear phase interface have been derived from the resultant balance laws of the general non-linear theory of shells. These balance equations at the quasistatically moving phase interface have been presented for thermoelastic and thermoviscoelastic two-phase materials. They generalize our earlier results obtained in [12, 42] by variational methods valid for the case of thermodynamic equilibrium state. In particular, we have used the kinetic equation (5.4) with the function (5.5) which allows

one to describe the motion of the curvilinear phase interface in the quasistatic deformation process.

To illustrate the 2D model we have solved analytically the simple 1D example of phase transition in the elastic circular cylindrical tube subjected to tensile forces applied to its end. For the linearly elastic cylinder one observes the existence of hysteresis loop characteristic to the behaviour of phase transitions in martensitic materials. The size of the loop depends upon the values of several loading parameters. This example contains, as particular cases, many existing 1D models of phase transformation in elastic bodies.

The proposed 2D model allows one to take into account several additional factors unavailable in the existing 1D models of phase transitions, such as solutions of the boundary layer type, or more differentiated ways of loading and unloading. We are also able to analyse, even analytically, quite complex problems which in the 3D models are possible to discuss only by numerical methods.

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