On modified displacement version of the non–linear theory of thin shells

S. Opoka[a], W. Pietraszkiewicz[a,*]

[a]*Institute of Fluid-Flow Machinery, PASci, ul. Fiszera 14, 80-952 Gdańsk, Poland*

Abstract

We discuss the non-linear theory of thin shells expressed in terms of displacements of the shell reference surface as the only independent field variables. The formulation is based on the principle of virtual work postulated for the reference surface. In our approach: 1) the vector equilibrium equations are represented through components in the deformed contravariant surface base, and using the compatibility conditions the resulting tangential equilibrium equations are additionally simplified, 2) at the shell boundary the new scalar function of displacement derivatives is defined and new sets of four work-conjugate static and geometric boundary conditions are derived, as well as 3) for prescribed shell geometry all non-linear shell relations are generated automatically by two packages set up in MATHEMATICA. The displacement boundary value problem (BVP) and associated homogeneous shell buckling problem are generated exactly without using any additional approximations following from errors of the constitutive equations. Both problems are extremely complex and available only in the computer memory. Such an approach allows us to account also for those a few supposedly small terms which may be critical for finding the correct buckling load of shells sensitive to imperfections. This approach is used in the accompanying paper by Opoka and Pietraszkiewicz (2009, submitted to Int. J. Solids Str.) to perform the refined numerical analysis of bifurcation buckling for the axially compressed circular cylinder.

Key words: Thin shell, Non-linear theory, Equilibrium equations, Boundary conditions, Buckling equations

---

*Corresponding author. Phone: +48586995263; Fax: +48583416144; E-mail: pietrasz@imp.gda.pl

Email addresses: `szop@imp.gda.pl` (S. Opoka)

Preprint submitted to International Journal of Solids and Structures *November 25, 2008*

1. Introduction

The entirely Lagrangian non-linear theory of thin elastic shells, expressed in terms of displacements $\mathbf{u}$ as the only independent field variables, was proposed by Pietraszkiewicz and Szwabowicz (1981) and developed by Pietraszkiewicz (1984), where references to earlier attempts in the field were given. The formulation was based on the principle of virtual work postulated for the shell reference surface. The resulting vector equilibrium equations as well as work-conjugate geometric and static boundary conditions were represented in components relative to the covariant base vectors $\mathbf{a}_\alpha$, $\mathbf{n}$ of the undeformed reference surface. Unfortunately, the resulting three scalar equilibrium equations and two sets of four work-conjugate scalar static and geometric boundary conditions of such a shell theory became very complex and hardly manageable.

Based on the displacement formulation of Pietraszkiewicz (1984) the extensive numerical tests of axisymmetric deflections and stability of thin shells of revolution undergoing large rotations were performed by Nolte (1983) and Nolte et al. (1986) as well as of rubberlike shells undergoing large strains by Schieck et al. (1992). In those papers the finite element method was applied with corresponding C^1 elements. The analysis revealed that the important part of complexities of such scalar equilibrium equations was associated with representing the covariant base vectors $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ of the deformed reference surface through those $\mathbf{a}_\alpha$, $\mathbf{n}$ of the undeformed reference surface and the displacement gradients $\mathbf{u}_{,\alpha}$. Also the use of scalar function n_ν of displacement derivatives at the shell boundary was found to be inconvenient in those numerical applications because of square-root functions of displacement derivatives appearing in denominators of static boundary quantities. Those conclusions and our recent experience gained while writing two reports by Opoka and Pietraszkiewicz (2004, 2009) allow us to propose in this paper the following three modifications of the non-linear displacement formulation of shell equations:

1. The vector equilibrium equations of Pietraszkiewicz (1984) are represented through components in the contravariant base $\bar{\mathbf{a}}^\alpha$, $\bar{\mathbf{n}}$ of the deformed reference surface, and the tangential scalar equilibrium equations are additionally exactly simplified using the compatibility conditions.
2. Along the boundary contour of the reference surface the new scalar function α rational with regard to displacement derivatives is defined and new sets of four work-conjugate static and geometric boundary conditions are derived.
3. For any definite geometry of the reference surface parametrized by orthogonal coordinates and for any of its boundaries, the displacement scalar equi-

librium equations and boundary conditions as well as the corresponding incremental displacement buckling shell equations and boundary conditions are generated automatically by the use of two packages *ShellGeom.m* and *ShellBVP.m* set up in MATHEMATICA.

The additional difference between our modified displacement shell BVP and other ones known in the literature is that we do not simplify the shell relations in the process of expressing the surface stress and strain measures in terms of displacements. As a result, the displacement BVP and associated buckling shell problem become extremely complex and not tractable by hand transformations. But thanks to the symbolic language of MATHEMATICA the complex shell equations become manageable as relations generated directly in the computer memory. The idea behind such a seemingly absurdal approach has been the necessity to account for those a few supposedly small terms in the buckling shell equations which may be critically important for finding the correct buckling load of shell structures sensitive to imperfections. In such problems small terms in the buckling shell equations play the role of some kind of imperfections.

The literature on the non-linear theory of shells provides many different suggestions how to reasonably simplify the complex displacement BVP and the stability problem. For example, within the first-approximation geometrically non-linear theory of thin, isotropic, elastic shells summarised by Pietraszkiewicz (1989) the most popular approach is to use explicitly known errors of the constitutive equations. When the surface stress measures are eliminated from the BVP and the stability problem, many supposedly small terms of the order of errors in the constitutive equations are omitted as well, see for example Koiter and Simmonds (1973) or Opoka and Pietraszkiewicz (2004). One can additionally restrict the order of allowable rotations expressed in powers of the error of the constitutive equations, and within various restricted versions of shell theory omit many other supposedly small terms, see for example Pietraszkiewicz (1984). Additionally simplified relations within the moderate rotation variant of shell equations were reviewed by Schmidt and Pietraszkiewicz (1981). One can also use the asymptotic methods relative to fractional powers of the error in the constitutive equations. Predicting the asymptotic behaviour of solution of a special shell problem one can omit many supposedly small terms as well and get very simple non-linear shell equations modelling this special problem, see for example Tovstik and Smirnov (2001).

In all such heuristic type of simplifications mentioned above it is implicitly assumed that omission of supposedly small terms in the shell equations does not have a significant influence on their solutions. However, this argument may not be correct when the non-linear shell relations are used to formulate and analyse the stability problem of thin shells sensitive to imperfections. In such a case omission of some supposedly small terms from the equilibrium equations may sometimes lead to a significantly different buckling load. For example Koiter (1960, Appendix) discovered that in analysis of the axially compressed circular cylinder with relaxed boundary conditions in the circumferential direction the omission of one supposedly small term in the linear equations of neutral equilibrium led to the incorrect buckling load for short cylinders. Also Opoka and Pietraszkiewicz (2009) explicitly show that omission in the non–linear BVP of all small terms of the order of the error introduced by the constitutive equations leads to overestimated buckling loads for long axially compressed cylinders. But we are not aware of any general method how one might discover such a few supposedly small but important terms and take them into account in a particular shell buckling problem, omitting at the same time many other insignificant terms of comparable order.

The paper is organized as follows. In Section 2 we remind basic notation and exact kinematic relations of the surface deformation. Modified equilibrium equations are derived in Section 3 by postulating the principle of virtual work (PVW) for the shell reference surface. Then in Section 4 we derive modified work-conjugate set of boundary conditions expressed through the new boundary function α describing the rotational deformation of the shell lateral boundary surface. The use of packages *ShellGeom.m* and *ShellBVP.m* for formulation of the modified BVP and the corresponding shell buckling problem in terms of displacement variables is described in Section 5, where some remarks on stability analysis are given.

2. Notation and kinematic relations

Let $\mathscr{P}$ be a region of the three-dimensional Euclidean point space $\mathscr{E}$ occupied by the shell in its undeformed configuration. In $\mathscr{P}$ we introduce the normal system of curvilinear coordinates $(\theta^1, \theta^2, \zeta)$ such that $-h/2 \leqslant \zeta \leqslant h/2$ is the distance from the middle surface $\mathscr{M}$ to points in $\mathscr{P}$, and h is the undeformed shell thickness assumed to be constant and small as compared with other shell dimensions and with the smallest radius of curvature of $\mathscr{M}$. The surface $\mathscr{M}$ is described by the position vector $\mathbf{r} = \mathbf{r}(\theta^\alpha)$ relative to a point $O \in \mathscr{E}$.

With each point $M \in \mathscr{M}$ we associate the natural covariant base vectors $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$, where comma denotes partial differentiation with respect to θ^α, the covariant $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ and the contravariant $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$ components of the surface metric tensor $\mathbf{a}$ with $a = \det(a_{\alpha\beta}) > 0$, the contravariant components $\varepsilon^{\alpha\beta}$ of the surface permutation tensor $\boldsymbol{\varepsilon}$ such that $\varepsilon^{12} = -\varepsilon^{21} = 1/\sqrt{a}$, $\varepsilon^{11} = \varepsilon^{22} = 0$, the unit normal vector $\mathbf{n} = \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ orienting $\mathscr{M}$, and the covariant components $b_{\alpha\beta} = -\mathbf{a}_{,\alpha} \cdot \mathbf{n}_{,\beta} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta}$ of the surface curvature tensor $\mathbf{b}$. The contravariant components $a^{\alpha\beta}$ of $\mathbf{a}$ satisfying the relations $a^{\alpha\gamma}a_{\beta\gamma} = \delta^\alpha_\beta$ are used to raise indices of components of the surface vectors and tensors. The natural connection on $\mathscr{M}$ is defined by the surface Christoffel symbols $\Gamma^\mu_{\alpha\beta} = \mathbf{a}^\mu \cdot \mathbf{a}_{\alpha,\beta}$.

The boundary contour $\partial\mathscr{M}$ of $\mathscr{M}$ consists of a finite number of piecewise smooth curves given by $\mathbf{r}(s) = \mathbf{r}[\theta(s)]$, where s is the arc-length along $\partial\mathscr{M}$. With each regular point $M \in \partial\mathscr{M}$ we associate the unit tangent vector $\boldsymbol{\tau} \equiv \mathbf{r}_{,s} = d\mathbf{r}/ds = \tau^\alpha \mathbf{a}_\alpha$, and the outward unit normal vector $\boldsymbol{\nu} \equiv \mathbf{r}_{,\nu} = d\mathbf{r}/ds_\nu = \boldsymbol{\tau} \times \mathbf{n} = \nu^\alpha \mathbf{a}_\alpha$, where s_ν is the arc-length of the surface curve perpendicular to the boundary contour in the outward normal direction. The curvature properties of $\partial\mathscr{M}$ are described by the normal curvature $\sigma_\tau = b_{\alpha\beta}\tau^\alpha\tau^\beta$, the geodesic torsion $\tau_\tau = -b_{\alpha\beta}\nu^\alpha\tau^\beta$, and the geodesic curvature $\rho_\tau = \nu^\alpha|_\alpha = \tau_\alpha \nu^\alpha|_\beta\, \tau^\beta = -\nu_\alpha \tau^\alpha|_\beta\, \tau^\beta$, where $(\ )|_\alpha$ denotes the covariant surface derivative with respect to curvilinear coordinates θ^α. The symbols $\sigma_\nu = b_{\alpha\beta}\nu^\alpha\nu^\beta$ and $\rho_\nu = \tau^\alpha|_\alpha = \nu_\alpha \tau^\alpha|_\beta\, \nu^\beta = -\tau_\alpha \nu^\alpha|_\beta\, \nu^\beta$ are the normal curvature and the geodesic curvature, respectively, of the surface curve orthogonal to $\partial\mathscr{M}$ in the outward normal direction. Physical components of surface tensors on $\partial\mathscr{M}$ are defined as, for example, $N_{\nu\nu} = N^{\alpha\beta}\nu_\alpha\nu_\beta$, $\kappa_{\nu\tau} = \kappa_{\alpha\beta}\nu^\alpha\tau^\beta$. For other geometric definitions and relations we refer to Green and Zerna (1968), Chernykh (1964), Flügge (1972), Pietraszkiewicz (1977) and Ciarlet (2005).

The deformed configuration $\bar{\mathscr{M}}$ of the surface $\mathscr{M}$ is described by the position vector $\bar{\mathbf{r}}(\theta^\alpha) = \mathbf{r}(\theta^\alpha)+\mathbf{u}(\theta^\alpha)$ relative to the same point $O \in \mathscr{E}$, where θ^α are the same surface curvilinear convected coordinates, and $\mathbf{u} = u^\alpha \mathbf{a}_\alpha + u^3\mathbf{n} = u_\alpha \mathbf{a}^\alpha + u_3\mathbf{n}$ is the displacement field. In convected coordinates geometric quantities and relations on the deformed surface $\bar{\mathscr{M}}$ are defined analogously as their counterparts in the undeformed configuration; they will be marked here by an additional dash, for example $\bar{\mathbf{a}}_\alpha$, $\bar{a}^{\alpha\beta}$, $\bar{b}_{\alpha\beta}$, $\bar{\mathbf{n}}$, $\bar{\boldsymbol{\nu}}$, $(\ )\|_\alpha$, etc. All dashed fields on $\bar{\mathscr{M}}$ can be expressed through analogous undashed fields defined on $\mathscr{M}$ and the displacement field $\mathbf{u}$, see for example Pietraszkiewicz (1984, 1989). In particular, we have

$$
\begin{aligned}
\bar{\mathbf{a}}_\alpha &= l_{\lambda\alpha}\mathbf{a}^\lambda + \varphi_\alpha \mathbf{n}\ , & \bar{\mathbf{n}} &= \sqrt{\tfrac{a}{\bar{a}}}\left(m_\lambda \mathbf{a}^\lambda + m\mathbf{n}\right) , \\
l_{\lambda\alpha} &= a_{\lambda\alpha} + u_\lambda|_\alpha - b_{\lambda\alpha}u_3\ , & \varphi_\alpha &= u_{3,\alpha} + b^\lambda_\alpha u_\lambda\ , \\
m_\lambda &= \varphi_\alpha l^\alpha_{.\lambda} - \varphi_\lambda l^\alpha_{.\alpha}\ , & m &= \tfrac{1}{2}\left(l^\alpha_{.\alpha} l^\beta_{.\beta} - l^\beta_{.\alpha} l^\alpha_{.\beta}\right) .
\end{aligned}
\tag{1}
$$

Components of the symmetric surface strain and bending measures of the

Green type are defined by the relations

$$\gamma_{\alpha\beta} = \tfrac{1}{2}\left(\bar{a}_{\alpha\beta} - a_{\alpha\beta}\right) \ , \quad \kappa_{\alpha\beta} = -\left(\bar{b}_{\alpha\beta} - b_{\alpha\beta}\right) \ . \tag{2}$$

Their expressions in terms of displacements following from (1) and (2) take the form

$$\begin{aligned}
\gamma_{\alpha\beta} &= \frac{1}{2}\left(\bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta - a_{\alpha\beta}\right) = \frac{1}{2}\left(l^\lambda_{.\alpha} l_{\lambda\beta} + \varphi_\alpha\varphi_\beta - a_{\alpha\beta}\right) \ , \\
\kappa_{\alpha\beta} &= b_{\alpha\beta} - \bar{\mathbf{n}} \cdot \bar{\mathbf{a}}_\alpha|_\beta = b_{\alpha\beta} - \sqrt{\tfrac{a}{\bar{a}}}\chi_{\alpha\beta} \ , \\
\chi_{\alpha\beta} &= m\left(\varphi_\alpha|_\beta + b_{\lambda\beta} l^\lambda_{.\alpha}\right) + m_\lambda\left(l^\lambda_{.\alpha}|_\beta - b^\lambda_\beta\varphi_\alpha\right) \ , \\
\frac{\bar{a}}{a} &= 1 + 2\gamma^\alpha_\alpha + 2\left(\gamma^\alpha_\alpha\gamma^\beta_\beta - \gamma^\beta_\alpha\gamma^\alpha_\beta\right) \ .
\end{aligned} \tag{3}$$

At the boundary contour $\partial\mathscr{M}$ the physical components of $(1)_2$ are

$$\begin{aligned}
&l_{\nu\nu} = 1 + u_\nu,_\nu + \rho_\nu u_\tau - \sigma_\nu u_3 \ , \quad l_{\nu\tau} = u_\nu,_s - \rho_\tau u_\tau + \tau_\tau u_3 \ , \quad m_\nu = \varphi_\tau l_{\tau\nu} - \varphi_\nu l_{\tau\tau} \ , \\
&l_{\tau\tau} = 1 + u_\tau,_s + \rho_\tau u_\nu - \sigma_\tau u_3 \ , \quad l_{\tau\nu} = u_\tau,_\nu - \rho_\nu u_\nu + \tau_\tau u_3 \ , \quad m_\tau = \varphi_\nu l_{\nu\tau} - \varphi_\tau l_{\nu\nu} \ , \\
&\varphi_\nu = u_3,_\nu + \sigma_\nu u_\nu - \tau_\tau u_\tau \ , \qquad \varphi_\tau = u_3,_s - \tau_\tau u_\nu + \sigma_\tau u_\tau \ , \quad m = l_{\nu\nu} l_{\tau\tau} - l_{\nu\tau} l_{\tau\nu} \ .
\end{aligned} \tag{4}$$

The deformed surface base $\{\bar{\mathbf{a}}_\alpha, \bar{\mathbf{n}}\}$ and the deformed boundary base $\{\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{\tau}}, \bar{\mathbf{n}}\}$ can be represented in the undeformed boundary base $\{\boldsymbol{\nu}, \boldsymbol{\tau}, \mathbf{n}\}$ as follows:

$$\begin{aligned}
&\bar{\mathbf{a}}_\alpha = \bar{\nu}_\alpha\bar{\boldsymbol{\nu}} + \bar{\tau}_\alpha\bar{\boldsymbol{\tau}} = l_{\lambda\alpha}\nu^\lambda\boldsymbol{\nu} + l_{\lambda\alpha}\tau^\lambda\boldsymbol{\tau} + \varphi_\alpha\mathbf{n} \ , \\
&\bar{\mathbf{n}} = \sqrt{\frac{a}{\bar{a}}}(m_\nu\boldsymbol{\nu} + m_\tau\boldsymbol{\tau} + m\mathbf{n}) = \sqrt{\frac{a}{\bar{a}}}\bar{\mathbf{m}} \ , \quad \bar{\mathbf{m}} = \bar{\mathbf{r}},_\nu \times \bar{\mathbf{r}},_s \ , \\
&\bar{\nu}^\alpha = \frac{1}{a_\tau}\sqrt{\frac{a}{\bar{a}}}(a_\tau^2\nu^\alpha - 2\gamma_{\tau\nu}\tau^\alpha) \ , \quad \bar{\tau}^\alpha = \frac{1}{a_\tau}\tau^\alpha \ , \quad a_\tau = \sqrt{1 + 2\gamma_{\tau\tau}} \ , \\
&\bar{\boldsymbol{\tau}} = \bar{\tau}^\alpha\bar{\mathbf{a}}_\alpha = \frac{1}{a_\tau}(l_{\nu\tau}\boldsymbol{\nu} + l_{\tau\tau}\boldsymbol{\tau} + \varphi_\tau\mathbf{n}) \ , \\
&\bar{\boldsymbol{\nu}} = \bar{\nu}^\alpha\bar{\mathbf{a}}_\alpha = \frac{1}{a_\tau}\sqrt{\frac{a}{\bar{a}}}\left[\,(l_{\tau\tau}m - \varphi_\tau m_\tau)\,\boldsymbol{\nu} + (\varphi_\tau m_\nu - l_{\nu\tau}m)\,\boldsymbol{\tau} + (l_{\nu\tau}m_\tau - l_{\tau\tau}m_\nu)\,\mathbf{n}\,\right] \ .
\end{aligned} \tag{5}$$

At the boundary contour $\partial\mathscr{M}$ the fields defined in (4) and (5) are again completely described by the geometry of $\mathscr{M}$ and the displacement components.

The surface strain and bending measures cannot be arbitrary functions of the surface coordinates. In order to represent deformation of the surface embedded in the three-dimensional Euclidean space they have to satisfy three differential conditions derived from the Codazzi-Mainardi-Gauss conditions for $\mathscr{M}$ and $\overline{\mathscr{M}}$. These exact compatibility conditions of the surface deformation are, see Koiter (1966),

$$\begin{aligned} &\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}\left[\kappa_{\beta\lambda}|_{\mu}+\overline{a}^{\kappa\rho}\left(b_{\kappa\lambda}-\kappa_{\kappa\lambda}\right)\gamma_{\rho\beta\mu}\right]=0\ , \\ &\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}\left[\gamma_{\alpha\mu}|_{\beta\lambda}-b_{\alpha\mu}\kappa_{\beta\lambda}+\tfrac{1}{2}\left(\kappa_{\alpha\mu}\kappa_{\beta\lambda}+\overline{a}^{\kappa\rho}\gamma_{\kappa\alpha\mu}\gamma_{\rho\beta\lambda}\right)\right]+K\gamma^{\kappa}_{\kappa}=0\ , \end{aligned} \tag{6}$$

where K is the Gaussian curvature of $\mathscr{M}$, and $\gamma_{\rho\beta\mu}=\gamma_{\rho\beta}|_{\mu}+\gamma_{\rho\mu}|_{\beta}-\gamma_{\beta\mu}|_{\rho}$.

3. Modified equilibrium equations

Under some kinematic assumptions summarised by Pietraszkiewicz (1989) for the geometrically non-linear theory of elastic shells and proposed by Schieck et al. (1992) for the large-strain theory of rubber-like shells, or alternately under the constitutive assumptions proposed by Libai and Simmonds (1998), the mechanical behaviour of a thin shell is entirely described by stretching and bending of its reference surface.

Let $\overline{\mathscr{M}}$ be the reference surface of the deformed shell in an equilibrium state under the surface force $\mathbf{p}(\theta^{\alpha})=p_{\alpha}\overline{\mathbf{a}}^{\alpha}+p\overline{\mathbf{n}}$ and couple $\mathbf{c}(\theta^{\alpha})=\overline{\mathbf{n}}\times c^{\alpha}\overline{\mathbf{a}}_{\alpha}$ vectors, both measured per unit area of the reference surface $\mathscr{M}$, and under the boundary force $\mathbf{N}^{*}(s)=N^{*}_{\nu}\boldsymbol{\nu}+N^{*}_{\tau}\boldsymbol{\tau}+N^{*}\mathbf{n}$ and couple $\mathbf{M}^{*}(s)=\overline{\mathbf{n}}\times(M^{*}_{\nu}\overline{\boldsymbol{\nu}}+M^{*}_{\tau}\overline{\boldsymbol{\tau}})$ vectors, both measured per unit length of the undeformed boundary contour $\partial\mathscr{M}$. Then, for all kinematically admissible virtual displacements $\delta\mathbf{u}$ the equilibrium conditions for $\overline{\mathscr{M}}$ are given by the principle of virtual work (PVW)

$$\iint\limits_{\mathscr{M}}\left(N^{\alpha\beta}\delta\gamma_{\alpha\beta}+M^{\alpha\beta}\delta\kappa_{\alpha\beta}\right)dA=\iint\limits_{\mathscr{M}}\left(\mathbf{p}\cdot\delta\mathbf{u}+\mathbf{c}\cdot\boldsymbol{\omega}\right)dA+\int\limits_{\partial\mathscr{M}_f}\left(\mathbf{N}^{*}\cdot\delta\mathbf{u}+\mathbf{M}^{*}\cdot\boldsymbol{\omega}_{\tau}\right)ds\ , \tag{7}$$

where $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are components of the symmetric surface stress resultants and stress couples of the Kirchhoff type, $\delta\gamma_{\alpha\beta}$ and $\delta\kappa_{\alpha\beta}$ are virtual changes of the strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$, while $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_{\tau}$ are the virtual rotation vectors at $\overline{\mathscr{M}}$ and along $\partial\overline{\mathscr{M}}$, respectively.

The strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ can also be represented in the hybrid form as

$$\gamma_{\alpha\beta}=\tfrac{1}{2}\left(\overline{\mathbf{r}},_{\alpha}\cdot\overline{\mathbf{r}},_{\beta}-a_{\alpha\beta}\right)\ ,\quad \kappa_{\alpha\beta}=\overline{\mathbf{r}},_{\alpha}\cdot\overline{\mathbf{n}},_{\beta}+b_{\alpha\beta}\ . \tag{8}$$

Hence, the integrand on the left–hand side of (7) can be given in the form

$$N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta} = \left(N^{\alpha\beta}\bar{\mathbf{a}}_\alpha + M^{\alpha\beta}\bar{\mathbf{n}},_\alpha\right)\cdot\delta\mathbf{u},_\beta + M^{\alpha\beta}\delta\bar{\mathbf{n}},_\beta\cdot\bar{\mathbf{a}}_\alpha \ . \tag{9}$$

The virtual rotation vector $\boldsymbol{\omega}$ is defined by

$$\omega = \frac{1}{2}\left(\bar{\mathbf{a}}^\alpha\times\delta\bar{\mathbf{a}}_\alpha + \bar{\mathbf{n}}\times\delta\bar{\mathbf{n}}\right) \ , \tag{10}$$

and the virtual work performed by $\mathbf{c}$ on $\boldsymbol{\omega}$ can equivalently be expressed as

$$\mathbf{c}\cdot\omega = \mathbf{h}\cdot\delta\bar{\mathbf{n}} \ , \quad \mathbf{h} = c^\alpha\bar{\mathbf{a}}_\alpha \ . \tag{11}$$

The vector $\mathbf{h}$ is usually called the surface static moment.

Inside $\mathscr{M}$, $\bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{n}} = 0$ which variated leads to

$$\delta\bar{\mathbf{n}} = -\bar{\mathbf{a}}^\beta\left(\bar{\mathbf{n}}\cdot\delta\mathbf{u},_\beta\right) \ . \tag{12}$$

Introducing (8) and (12) into (7) and using the Stokes theorem in $\mathscr{M}$, after some transformations we obtain

$$\begin{aligned}&-\iint\limits_{\mathscr{M}}\left[\mathbf{T}^\beta|_\beta + \mathbf{p} + \left(c^\beta\bar{\mathbf{n}}\right)|_\beta\right]\cdot\delta\mathbf{u}dA\\ &+\int\limits_{\partial\mathscr{M}}\left[\left(\mathbf{T}^\beta + c^\beta\bar{\mathbf{n}}\right)\nu_\beta\cdot\delta\mathbf{u} + M^{\alpha\beta}\nu_\beta\bar{\mathbf{a}}_\alpha\cdot\delta\bar{\mathbf{n}}\right]ds - \int\limits_{\partial\mathscr{M}_f}\left(\mathbf{N}^*\cdot\delta\mathbf{u} + \mathbf{M}^*\cdot\omega_\tau\right)ds = 0 \ ,\end{aligned} \tag{13}$$

where

$$\mathbf{T}^\beta = \left(N^{\alpha\beta} - \bar{b}^\alpha_\lambda M^{\lambda\beta}\right)\bar{\mathbf{a}}_\alpha + \left(M^{\alpha\beta}|_\alpha + \bar{a}^{\beta\kappa}\gamma_{\kappa\lambda\mu}M^{\lambda\mu}\right)\bar{\mathbf{n}} \ . \tag{14}$$

From vanishing of the surface integral in (13) follows the known vector equilibrium equation

$$\mathbf{T}^\beta|_\beta + \mathbf{p} + \left(c^\beta\bar{\mathbf{n}}\right)|_\beta = \mathbf{0} \quad \text{in } \mathscr{M} \ . \tag{15}$$

The vector equation (15) can be represented in different surface bases to obtain formally different but in fact equivalent sets of three scalar equilibrium equations reviewed by Pietraszkiewicz (1989). In particular, the component forms of (15) (without surface couple term $(c^\beta\bar{\mathbf{n}})|_\beta$) in the deformed covariant base $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ were given already by Chien (1944), Galimov (1951) and Danielson (1970). Different but equivalent component forms of (15) (again without $(c^\beta\bar{\mathbf{n}})|_\beta$) in the base $\mathbf{a}_\alpha$,

n given by Sanders (1963), Budiansky (1968) and Pietraszkiewicz (1974) were associated with different definitions of the bending tensors. However, our recent experience indicates that the simplest scalar equilibrium equations following from (15) are obtained in the deformed contravariant base $\bar{\mathbf{a}}^\alpha$, $\bar{\mathbf{n}}$.

Taking into account differential rules of the deformed base vectors

$$\bar{\mathbf{a}}_\alpha|_\beta = \bar{b}_{\alpha\beta}\bar{\mathbf{n}} + \gamma_{\lambda\alpha\beta}\bar{\mathbf{a}}^\lambda\ , \quad \bar{\mathbf{a}}^\alpha|_\beta = \bar{b}^\alpha_\beta\bar{\mathbf{n}} - \bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta}\bar{\mathbf{a}}^\lambda\ , \quad \bar{\mathbf{n}},_\alpha = \bar{\mathbf{n}}|_\alpha = -\bar{b}_{\alpha\lambda}\bar{\mathbf{a}}^\lambda\ , \tag{16}$$

from (15) in the base $\bar{\mathbf{a}}^\alpha$, $\bar{\mathbf{n}}$ we obtain three scalar equilibrium equations

$$\begin{aligned}
&\bar{a}_{\alpha\lambda}N^{\lambda\beta}|_\beta + \left(2\gamma_{\alpha\lambda}|_\beta - \gamma_{\lambda\beta}|_\alpha\right)N^{\lambda\beta} - \bar{b}_{\alpha\lambda}|_\beta\, M^{\lambda\beta} - 2\bar{b}_{\alpha\lambda}M^{\lambda\beta}|_\beta \\
&\qquad - \bar{a}^{\kappa\mu}\left(\bar{b}_{\alpha\kappa}\gamma_{\mu\lambda\beta} - \bar{b}_{\lambda\kappa}\gamma_{\mu\alpha\beta}\right)M^{\lambda\beta} - \bar{b}_{\alpha\lambda}c^\lambda + p_\alpha = 0\ , \\
&M^{\alpha\beta}|_{\alpha\beta} + \bar{b}_{\alpha\beta}N^{\alpha\beta} - \bar{a}^{\alpha\mu}\bar{b}_{\alpha\lambda}\bar{b}_{\mu\beta}M^{\lambda\beta} + \left[\bar{a}^{\alpha\mu}\left(2\gamma_{\mu\lambda}|_\beta - \gamma_{\lambda\beta}|_\mu\right)M^{\lambda\beta}\right]\Big|_\alpha + c^\alpha|_\alpha + p = 0\ .
\end{aligned} \tag{17}$$

But with the Codazzi-Mainardi relations $b_{\beta\lambda}|_\mu = b_{\beta\mu}|_\lambda$ for $\mathscr{M}$ the tangential compatibility conditions $(6)_1$ can be written as

$$\bar{\varepsilon}^{\lambda\mu}\left(-\bar{b}_{\lambda\beta}|_\mu + \bar{a}^{\kappa\rho}\bar{b}_{\lambda\kappa}\gamma_{\rho\mu\beta}\right) = 0\ , \tag{18}$$

or

$$\bar{b}_{\alpha\beta}|_\lambda - \bar{b}_{\lambda\beta}|_\alpha - \bar{a}^{\kappa\mu}\left(\bar{b}_{\alpha\kappa}\gamma_{\mu\lambda\beta} - \bar{b}_{\lambda\kappa}\gamma_{\mu\alpha\beta}\right) = 0\ , \tag{19}$$

so that the complex terms in $(17)_1$ can be exactly replaced by much simpler ones:

$$\bar{a}^{\kappa\mu}\left(\bar{b}_{\alpha\kappa}\gamma_{\mu\lambda\beta} - \bar{b}_{\lambda\kappa}\gamma_{\mu\alpha\beta}\right)M^{\lambda\beta} = \left(\bar{b}_{\alpha\beta}|_\lambda - \bar{b}_{\lambda\beta}|_\alpha\right)M^{\lambda\beta}\ . \tag{20}$$

Unfortunately, a similar procedure applied to some terms in the third equilibrium equation $(17)_2$ using the compatibility condition $(6)_2$ leads to a more complicated form of $(17)_2$.

Expressing $\bar{a}_{\alpha\lambda}$ and $\bar{b}_{\alpha\lambda}$ through $\gamma_{\alpha\lambda}$ and $\kappa_{\alpha\lambda}$ according to (3) and applying the kinematic relation $\bar{a}^{\alpha\mu} = \frac{a}{\bar{a}}\left[\left(1 + 2\gamma^\lambda_\lambda\right)a^{\alpha\mu} - 2\gamma^{\alpha\mu}\right]$ and (20) we obtain three scalar equilibrium equations expressed through the mixed components of the surface stress and strain measures

$$\begin{aligned}
&\underline{N^\beta_\alpha|_\beta} + 2\gamma^\lambda_\alpha N^\beta_\lambda|_\beta + \left(2\gamma^\lambda_\alpha|_\beta - \gamma^\lambda_\beta|_\alpha\right)N^\beta_\lambda - 2\left(b^\lambda_\alpha - \kappa^\lambda_\alpha\right)M^\beta_\lambda|_\beta \\
&\qquad + \left[\left(b^\lambda_\beta - \kappa^\lambda_\beta\right)|_\alpha - 2\left(b^\lambda_\alpha - \kappa^\lambda_\alpha\right)|_\beta\right]M^\beta_\lambda - \left(b^\lambda_\alpha - \kappa^\lambda_\alpha\right)c_\lambda + p_\alpha = 0\ , \\
&\underline{M^{\alpha}_{\beta}|^\beta_\alpha + \left(b^\beta_\alpha - \kappa^\beta_\alpha\right)N^\alpha_\beta} - \frac{a}{\bar{a}}\left[\left(1 + 2\gamma^\mu_\mu\right)\left(b^\alpha_\beta - \kappa^\alpha_\beta\right) - 2\gamma^\alpha_\mu\left(b^\mu_\beta - \kappa^\mu_\beta\right)\right]\left(b^\lambda_\alpha - \kappa^\lambda_\alpha\right)M^\beta_\lambda \\
&\qquad + \left\{\frac{a}{\bar{a}}\left[\left(1 + 2\gamma^\mu_\mu\right)\left(2\gamma^\alpha_\beta|^\lambda - \gamma^\lambda_\beta|^\alpha\right) - 2\gamma^\alpha_\mu\left(2\gamma^\mu_\beta|^\lambda - \gamma^\lambda_\beta|^\mu\right)\right]M^\beta_\lambda\right\}\Big|_\alpha + c^\alpha|_\alpha + p = 0\ .
\end{aligned} \tag{21}$$

The equilibrium equations (21) are two-dimensionally exact for the shell reference surface in the sense that no approximations are introduced into them beyond those included in the initially postulated form (7) of the PVW. Notice that the tangential equilibrium equations $(21)_1$ are here even simpler than the analogous approximate expressions in the refined intrinsic shell equations derived by Pietraszkiewicz (1980), eq. (4.4.7). In fact, the modified equilibrium equations (21) following exactly from the PVW (7) are possibly the simplest ones available in the literature. We shall use them in Section 5 to generate the displacement form of equilibrium equations.

4. Modified work-conjugate boundary conditions

The appropriate boundary and corner conditions to be used with the equilibrium equations (15) should follow from vanishing of the line integrals in (13).

Along $\partial\mathscr{M}$ the virtual rotation vector $\boldsymbol{\omega}_\tau$ is now defined by

$$\boldsymbol{\omega}_\tau = \frac{1}{2}\left(\bar{\boldsymbol{\nu}} \times \delta\bar{\boldsymbol{\nu}} + \bar{\boldsymbol{\tau}} \times \delta\bar{\boldsymbol{\tau}} + \bar{\mathbf{n}} \times \delta\bar{\mathbf{n}}\right) \;, \tag{22}$$

and the virtual work performed by $\mathbf{M}^*$ on $\boldsymbol{\omega}_\tau$ can equivalently be expressed as

$$\mathbf{M}^* \cdot \boldsymbol{\omega}_\tau = \mathbf{H}^* \cdot \delta\bar{\mathbf{n}} \,, \quad \mathbf{H}^* = M^*_\nu \bar{\boldsymbol{\nu}} + M^*_\tau \bar{\boldsymbol{\tau}} \;. \tag{23}$$

The vector $\mathbf{H}$ can be called the boundary static moment.

The relation (12) for $\delta\bar{\mathbf{n}}$, when written at $\partial\mathscr{M}$ reads

$$\delta\bar{\mathbf{n}} = -\nu_\beta \bar{\mathbf{a}}^\beta \left(\bar{\mathbf{n}} \cdot \delta\mathbf{u},_\nu\right) - \tau_\beta \bar{\mathbf{a}}^\beta \left(\bar{\mathbf{n}} \cdot \delta\mathbf{u},_s\right) \;. \tag{24}$$

The expression (24) might be substituted into the second term of the first line integral of (13) and all terms containing $\bar{\mathbf{n}} \cdot \delta\mathbf{u},_s$ might be eliminated by integration by parts. Unfortunately, the remaining term containing the differential one-form of displacement derivatives $\bar{\mathbf{n}} \cdot \delta\mathbf{u},_\nu$ was proved by Makowski and Pietraszkiewicz (1989) to be neither exact nor integrable in terms of displacement (or position) derivatives along $\partial\mathscr{M}$. This means that there is no function $\phi(\mathbf{u},_\nu, \mathbf{u},_s)$ such that $\mu\bar{\mathbf{n}} \cdot \delta\mathbf{u},_\nu = \delta\phi$, where $\mu(\mathbf{u},_\nu, \mathbf{u},_s)$ is an integrating factor.

In order to derive appropriate work-conjugate sets of boundary conditions one has to modify the relation (24) to make it expressible through variation of a scalar function of displacement derivatives and $\bar{\mathbf{n}} \cdot \delta\mathbf{u},_s$ at $\partial\mathscr{M}$. As it was found by Makowski and Pietraszkiewicz (1989), there were three such scalar functions available in the literature: $n_\nu = \bar{\mathbf{n}} \cdot \boldsymbol{\nu}$ proposed by Pietraszkiewicz and Szwabowicz (1981), $\vartheta_\nu = (\bar{\mathbf{n}} - \mathbf{n}) \cdot \bar{\mathbf{a}}_\nu / a_\tau^2$ proposed by Novozhilov and Shamina (1975), and the angle ω_t of total rotation of the shell lateral boundary surface, see Pietraszkiewicz (1993). The fourth function θ was proposed by Libai and Simmonds (1998). However, the functions n_ν, ϑ_ν , θ (or ω_t) generate at $\partial\mathscr{M}$ the static fields containing the square-root (or trigonometric) functions of displacement derivatives in the denominator. Such expressions of the static boundary conditions are inconvenient in derivation of the stability problem and subsequent numerical analysis.

From our numerical experience gained in the recent report by Opoka and Pietraszkiewicz (2009), in this paper we introduce along $\partial\mathscr{M}$ the new scalar function of displacement (or position) derivatives defined by

$$\alpha = \frac{n_\nu}{n} = \frac{m_\nu}{m} = \frac{\varphi_\tau l_{\tau\nu} - \varphi_\nu l_{\tau\tau}}{l_{\nu\nu} l_{\tau\tau} - l_{\nu\tau} l_{\tau\nu}} = \alpha(\mathbf{u},_\nu , \mathbf{u},_s) \ . \tag{25}$$

The function α is rational one in terms of displacement derivatives, it vanishes in the undeformed state and upon linearization coincides with $-\varphi_\nu = -\mathbf{n} \cdot \mathbf{u},_\nu$, which is the infinitesimal rotation about tangent to $\partial\mathscr{M}$ used in the classical linear theory of shells.

Along $\partial\mathscr{M}$ the vector $\bar{\mathbf{n}}$ satisfies two constraints

$$\bar{\mathbf{r}},_s \cdot \bar{\mathbf{n}} = 0 \ , \quad \bar{\mathbf{n}} \cdot \bar{\mathbf{n}} = 1 \ . \tag{26}$$

From $(5)_1$ and (25) it follows that the constraint $(26)_1$ leads to the relation

$$\frac{m_\tau}{m} = -\frac{1}{l_{\tau\tau}}(l_{\nu\tau}\alpha + \varphi_\tau) \ , \tag{27}$$

and then the constraint $(26)_2$ allows one to represent m entirely in terms of α and $\mathbf{u},_s$:

$$m = \pm\sqrt{\frac{\bar{a}}{a}} \frac{l_{\tau\tau}}{\sqrt{l_{\tau\tau}^2(1+\alpha^2) + (l_{\nu\tau}\alpha + \varphi_\tau)^2}} = m(\alpha, \mathbf{u},_s) \ . \tag{28}$$

For rotations of the shell lateral boundary surface not exceeding $\pm\pi/2$ the + sign should be taken in (28). Hence, the formula for $\bar{\mathbf{n}}$ in terms of α and $\mathbf{u},_s$ becomes

$$\bar{\mathbf{n}} = \sqrt{\frac{a}{\bar{a}}} m \left[\alpha\boldsymbol{\nu} - \frac{1}{l_{\tau\tau}}(l_{\nu\tau}\alpha + \varphi_\tau)\boldsymbol{\tau} + \mathbf{n} \right] \ . \tag{29}$$

Let us variate the expression (25) to have

$$\begin{aligned}\delta\alpha &= \frac{1}{m^2}(m\delta m_\nu - m_\nu\delta m) = \frac{1}{m^2}[m\delta(\bar{\mathbf{r}},_\nu \times \bar{\mathbf{r}},_s) \cdot \boldsymbol{\nu} - m_\nu \delta(\bar{\mathbf{r}},_\nu \times \bar{\mathbf{r}},_s) \cdot \mathbf{n}] \\ &= \frac{1}{m}\{[\bar{\mathbf{r}},_s \times (\boldsymbol{\nu} - \alpha\mathbf{n})] \cdot \delta\mathbf{u},_\nu - [\bar{\mathbf{r}},_\nu \times (\boldsymbol{\nu} - \alpha\mathbf{n})] \cdot \delta\mathbf{u},_s\} \\ &= \frac{1}{m^2}(l_{\tau\nu}\bar{\mathbf{m}} \cdot \delta\mathbf{u},_s - l_{\tau\tau}\bar{\mathbf{m}} \cdot \delta\mathbf{u},_\nu) ,\end{aligned} \tag{30}$$

where we have used the identity

$$\frac{1}{l_{\tau\tau}}(\alpha l_{\nu\tau} + \varphi_\tau) = \frac{1}{l_{\tau\nu}}(\alpha l_{\nu\nu} + \varphi_\nu) \tag{31}$$

following from definitions (4). Calculating $\bar{\mathbf{m}} \cdot \delta\mathbf{u},_\nu$ from (30) and substituting it into (24) we obtain the new relation for $\delta\bar{\mathbf{n}}$ at $\partial\mathscr{M}$

$$\delta\bar{\mathbf{n}} = \bar{\mathbf{a}}^\beta \left[\sqrt{\frac{a}{\bar{a}}}\frac{m^2}{l_{\tau\tau}}\nu_\beta\delta\alpha - \left(\frac{l_{\tau\nu}}{l_{\tau\tau}}\nu_\beta + \tau_\beta\right)\bar{\mathbf{n}} \cdot \delta\mathbf{u},_s \right] . \tag{32}$$

The expressions (32) and (23) can now be used in the second terms of the line integrals in (13). After integration by parts and some transformations the line integral becomes

$$\begin{aligned}&\int_{\partial\mathscr{M}_f}\Big\{[C^\alpha\bar{\mathbf{a}}_\alpha + \mathscr{D}\bar{\mathbf{n}} - (\mathbf{N}^* - \bar{a}^{\alpha\mu}\bar{b}_{\mu\lambda}\tau^\lambda\mathscr{F}^*\bar{\mathbf{a}}_\alpha + \mathscr{F}^*,_s\bar{\mathbf{n}})] \cdot \delta\mathbf{u} \\ &+ \sqrt{\tfrac{a}{\bar{a}}}\tfrac{m^2}{l_{\tau\tau}}\left(M_{\nu\nu} - a_\tau\sqrt{\tfrac{\bar{a}}{a}}M^*_\nu\right)\delta\alpha\Big\}\,ds + \sum_{C_i\in\partial\mathscr{M}_f}(\mathscr{F} - \mathscr{F}^*)\bar{\mathbf{n}} \cdot \delta\mathbf{u}\Big|_{C_i^-}^{C_i^+} \\ &+ \int_{\partial\mathscr{M}_d}\Big[(C^\alpha\bar{\mathbf{a}}_\alpha + \mathscr{D}\bar{\mathbf{n}}) \cdot \delta\mathbf{u} + \sqrt{\tfrac{a}{\bar{a}}}\tfrac{m^2}{l_{\tau\tau}}M_{\nu\nu}\delta\alpha\Big]\,ds + \sum_{C_i\in\partial\mathscr{M}_d}\mathscr{F}\bar{\mathbf{n}} \cdot \delta\mathbf{u}\Big|_{C_i^-}^{C_i^+} = 0 ,\end{aligned} \tag{33}$$

where $\partial\mathscr{M}_d = \partial\mathscr{M}\backslash\partial\mathscr{M}_f$, C_i are the corner points of $\partial\mathscr{M}$, and

$$\begin{aligned}C^\alpha &= \underline{N^{\alpha\beta}\nu_\beta} - \bar{a}^{\alpha\mu}\bar{b}_{\mu\lambda}\left(M^{\lambda\beta}\nu_\beta + \tau^\lambda\mathscr{F}\right) , \\ \mathscr{D} &= \underline{M^{\alpha\beta}|_\beta\,\nu_\alpha + \mathscr{F},_s} + \bar{a}^{\alpha\mu}\gamma_{\mu\lambda\beta}M^{\lambda\beta}\nu_\alpha + c^\alpha\nu_\alpha , \\ \mathscr{F} &= \frac{l_{\tau\nu}}{l_{\tau\tau}}M_{\nu\nu} + M_{\nu\tau} , \\ \mathscr{F}^* &= \frac{1}{a_\tau}\left(M^*_\tau - \sqrt{\tfrac{\bar{a}}{a}}\tfrac{l_{\nu\tau}m - \varphi_\tau m_\nu}{l_{\tau\tau}}M^*_\nu\right) .\end{aligned} \tag{34}$$

All fields present in the boundary conditions (33) and (34) are functions of the arc-length coordinate s of $\partial\mathscr{M}$.

From (33) follow the natural static boundary and corner conditions

$$C^{\alpha}\bar{\mathbf{a}}_{\alpha}+\mathcal{D}\bar{\mathbf{n}}=\mathbf{N}^{*}-\bar{a}^{\alpha\mu}\bar{b}_{\mu\lambda}\tau^{\lambda}\mathcal{F}^{*}\bar{\mathbf{a}}_{\alpha}+\mathcal{F}^{*},_{s}\bar{\mathbf{n}}\,,\qquad M_{\nu\nu}=a_{\tau}\sqrt{\tfrac{a}{\bar{a}}}M_{\nu}^{*}\ \text{ on }\partial\mathscr{M}_{f}\,,\quad(35)$$

$$\mathcal{F}\bar{\mathbf{n}}=\mathcal{F}^{*}\bar{\mathbf{n}}\ \text{ at each corner }C_{i}\in\partial\mathscr{M}_{f}\,.\qquad(36)$$

The vector $\mathbf{N}^{*}-\bar{a}^{\alpha\mu}\bar{b}_{\mu\lambda}\tau^{\lambda}\mathcal{F}^{*}\bar{\mathbf{a}}_{\alpha}+\mathcal{F}^{*},_{s}\bar{\mathbf{n}}$ can be called the effective Kirchhoff stress resultant associated with the function α.

From (33) it also follows that the geometric boundary conditions which are work-conjugate to the static ones (35) are

$$\mathbf{u}=\mathbf{u}^{*},\qquad\alpha(\mathbf{u},_{\nu},\mathbf{u},_{s})=\alpha^{*}\ \text{ on }\partial\mathscr{M}_{d}\,.\qquad(37)$$

The last term of (33) vanishes identically at any corner $C_i\in\partial\mathscr{M}_d$, because deformation of the shell reference surface is assumed to be continuous everywhere including corners of the boundary contour $\partial\mathscr{M}$.

It is apparent from (33) and (29) that the displacement vector $\mathbf{u}$ on $\mathscr{M}$ is kinematically admissible if $\delta\mathbf{u}\equiv\delta\bar{\mathbf{r}}=\mathbf{0}$ and $\delta\alpha=0$ on $\partial\mathscr{M}_d$. Hence, $\delta\bar{\mathbf{n}}=\mathbf{0}$ on $\partial\mathscr{M}_d$ as well.

The vector static boundary and corner conditions (35) and (36) are direct and exact implication of the PVW (7) as well as the choice of α for description of rotation of the shell lateral boundary surface. Alternative vector static boundary conditions associated with n_ν were given by Pietraszkiewicz (1984), those compatible with ϑ_ν by Makowski and Pietraszkiewicz (1989), the ones compatible with ω_t by Pietraszkiewicz (1993), and those associated with θ by Libai and Simmonds (1998). Each of them may be more convenient than others in specific applications. In the refined numerical analysis of bifurcation buckling for the axially compressed circular cylinder performed by Opoka and Pietraszkiewicz (2009) we have found the choice of α with the corresponding boundary conditions (35)-(37) to be more convenient than other possible choices.

When the shell equations are used in the numerical analysis, we have to represent the boundary conditions (35), (36) and (37) through components in the undeformed boundary base $\{\boldsymbol{\nu},\boldsymbol{\tau},\mathbf{n}\}$. The reason for this choice is that the geometry of $\partial\mathscr{M}$ is the only one known in advance. With this choice it is also necessary to express the corresponding compound fields $C_\nu=C^\alpha\nu_\alpha$, $C_\tau=C^\alpha\tau_\alpha$, $\mathcal{D}$ in terms of physical components of the strain and stress measures at $\partial\mathscr{M}$. Thus, after some transformations we obtain

$$
\begin{aligned}
\mathcal{C}_\nu =& \underline{N_{\nu\nu}} - \frac{a}{\underline{a}}\Big[(1+2\gamma_{\tau\tau})(\sigma_\nu-\kappa_{\nu\nu}) + 2\gamma_{\nu\tau}(\tau_\tau+\kappa_{\nu\tau})\Big]M_{\nu\nu} \\
& + \frac{a}{\underline{a}}\Big[(1+2\gamma_{\tau\tau})(\tau_\tau+\kappa_{\nu\tau}) + 2\gamma_{\nu\tau}(\sigma_\tau-\kappa_{\tau\tau})\Big]\left(\frac{l_{\tau\nu}}{l_{\tau\tau}}M_{\nu\nu} + 2M_{\nu\tau}\right) , \\
\mathcal{C}_\tau =& \underline{N_{\nu\tau}} + \frac{a}{\underline{a}}\Big[(1+2\gamma_{\nu\nu})(\tau_\tau+\kappa_{\nu\tau}) + 2\gamma_{\nu\tau}(\sigma_\nu-\kappa_{\nu\nu})\Big]M_{\nu\nu} \\
& - \frac{a}{\underline{a}}\Big[(1+2\gamma_{\nu\nu})(\sigma_\tau-\kappa_{\tau\tau}) + 2\gamma_{\nu\tau}(\tau_\tau+\kappa_{\nu\tau})\Big]\left(\frac{l_{\tau\nu}}{l_{\tau\tau}}M_{\nu\nu} + 2M_{\nu\tau}\right) , \\
\mathcal{D} =& \underline{M_{\nu\nu},_\nu + 2M_{\nu\tau},_s + \rho_\tau(M_{\nu\nu}-M_{\tau\tau}) + 2\rho_\nu M_{\nu\tau} + \frac{1}{l_{\tau\tau}}(l_{\tau\nu}M_{\nu\nu}),_s - \frac{l_{\tau\nu}}{l_{\tau\tau}^2}l_{\tau\tau},_s M_{\nu\nu} + c_\nu} \\
& + \frac{a}{\underline{a}}\Big\{(1+2\gamma_{\tau\tau})(\gamma_{\nu\nu\nu}M_{\nu\nu} + 2\gamma_{\nu\nu\tau}M_{\nu\tau} + \gamma_{\nu\tau\tau}M_{\tau\tau}) - 2\gamma_{\nu\tau}(\gamma_{\tau\nu\nu}M_{\nu\nu} + 2\gamma_{\tau\nu\tau}M_{\nu\tau} + \gamma_{\tau\tau\tau}M_{\tau\tau})\Big\} ,
\end{aligned}
\tag{38}
$$

where the physical components of $\gamma_{\lambda\alpha\beta}$ at $\partial\mathscr{M}$ in $(38)_3$ are given by

$$
\begin{aligned}
&\gamma_{\nu\nu\nu} = \gamma_{\nu\nu},_\nu + 2\rho_\nu\gamma_{\nu\tau} , \quad \gamma_{\nu\nu\tau} = \gamma_{\nu\tau\nu} = \gamma_{\nu\nu},_s - 2\rho_\tau\gamma_{\nu\tau} , \\
&\gamma_{\nu\tau\tau} = 2\gamma_{\nu\tau},_s - \gamma_{\tau\tau},_\nu + 2\rho_\nu\gamma_{\nu\tau} + 2\rho_\tau(\gamma_{\nu\nu}-\gamma_{\tau\tau}) , \\
&\gamma_{\tau\nu\nu} = 2\gamma_{\nu\tau},_\nu - \gamma_{\nu\nu},_s + 2\rho_\tau\gamma_{\nu\tau} - 2\rho_\nu(\gamma_{\nu\nu}-\gamma_{\tau\tau}) , \\
&\gamma_{\tau\nu\tau} = \gamma_{\tau\tau\nu} = \gamma_{\tau\tau},_\nu - 2\rho_\nu\gamma_{\nu\tau} , \quad \gamma_{\tau\tau\tau} = \gamma_{\tau\tau},_s + 2\rho_\tau\gamma_{\nu\tau} .
\end{aligned}
\tag{39}
$$

Substituting (39) into $(38)_3$ the expression for $\mathcal{D}$ reads

$$
\begin{aligned}
\mathcal{D} =& \underline{M_{\nu\nu},_\nu + 2M_{\nu\tau},_s + \rho_\tau(M_{\nu\nu}-M_{\tau\tau}) + 2\rho_\nu M_{\nu\tau} + \frac{1}{l_{\tau\tau}}(l_{\tau\nu}M_{\nu\nu}),_s - \frac{l_{\tau\nu}}{l_{\tau\tau}^2}l_{\tau\tau},_s M_{\nu\nu} + c_\nu} \\
& + \frac{a}{\underline{a}}\Big\{[(1+2\gamma_{\tau\tau})\gamma_{\nu\nu},_\nu - 2\gamma_{\nu\tau}(2\gamma_{\nu\tau},_\nu - \gamma_{\nu\nu},_s - \rho_\nu(1+2\gamma_{\nu\nu}) + 2\rho_\tau\gamma_{\nu\tau})]M_{\nu\nu} \\
& + 2[(1+2\gamma_{\tau\tau})(\gamma_{\nu\nu},_s - 2\rho_\tau\gamma_{\nu\tau}) - 2\gamma_{\nu\tau}(\gamma_{\tau\tau},_\nu - 2\rho_\nu\gamma_{\nu\tau})]M_{\nu\tau} \\
& + [(1+2\gamma_{\tau\tau})(2\gamma_{\nu\tau},_s - \gamma_{\tau\tau},_\nu + 2\rho_\nu\gamma_{\nu\tau} + 2\rho_\tau(\gamma_{\nu\nu}-\gamma_{\tau\tau})) - 2\gamma_{\nu\tau}(\gamma_{\tau\tau},_s + 2\rho_\tau\gamma_{\nu\tau})]M_{\tau\tau}\Big\} .
\end{aligned}
\tag{40}
$$

Hence, the final scalar forms of four work-conjugate static and geometric boundary conditions along $\partial\mathscr{M}$ are

$$
\begin{aligned}
&l_{\nu\nu}C_\nu + l_{\nu\tau}C_\tau + m_\nu\sqrt{\tfrac{\bar{a}}{a}}\mathcal{D} = N^*_\nu + l_{\nu\nu}\mathcal{G}^*_\nu + l_{\nu\tau}\mathcal{G}^*_\tau + m_\nu\sqrt{\tfrac{\bar{a}}{a}}\mathcal{F}^*,_s \quad \text{or} \quad u_\nu = u^*_\nu\ ,\\
&l_{\tau\nu}C_\nu + l_{\tau\tau}C_\tau + m_\tau\sqrt{\tfrac{\bar{a}}{a}}\mathcal{D} = N^*_\tau + l_{\tau\nu}\mathcal{G}^*_\nu + l_{\tau\tau}\mathcal{G}^*_\tau + m_\tau\sqrt{\tfrac{\bar{a}}{a}}\mathcal{F}^*,_s \quad \text{or} \quad u_\tau = u^*_\tau\ ,\\
&\varphi_\nu C_\nu + \varphi_\tau C_\tau + m\sqrt{\tfrac{\bar{a}}{a}}\mathcal{D} = N^* + \varphi_\nu\mathcal{G}^*_\nu + \varphi_\tau\mathcal{G}^*_\tau + m\sqrt{\tfrac{\bar{a}}{a}}\mathcal{F}^*,_s \quad \text{or} \quad u_3 = u^*_3\ ,\\
&M_{\nu\nu} = a_\tau\sqrt{\tfrac{\bar{a}}{a}}M^*_\nu \quad \text{or} \quad \alpha = \alpha^*\ ,
\end{aligned}
\tag{41}
$$

where the expressions $\mathcal{G}^*_\nu, \mathcal{G}^*_\tau, \mathcal{F}^*,_s$ containing the external boundary moments are defined as follows:

$$
\begin{aligned}
\mathcal{G}^*_\nu &= \frac{1}{a_\tau}\frac{a}{\bar{a}}\Big[(1+2\gamma_{\tau\tau})(\tau_\tau+\kappa_{\nu\tau}) + 2\gamma_{\nu\tau}(\sigma_\tau-\kappa_{\tau\tau})\Big]\left(M^*_\tau - \sqrt{\frac{a}{\bar{a}}}\frac{l_{\nu\tau}m-\varphi_\tau m_\nu}{l_{\tau\tau}}M^*_\nu\right)\ ,\\
\mathcal{G}^*_\tau &= \frac{1}{a_\tau}\frac{a}{\bar{a}}\Big[(1+2\gamma_{\nu\nu})(\kappa_{\tau\tau}-\sigma_\tau) - 2\gamma_{\nu\tau}(\tau_\tau+\kappa_{\nu\tau})\Big]\left(M^*_\tau - \sqrt{\frac{a}{\bar{a}}}\frac{l_{\nu\tau}m-\varphi_\tau m_\nu}{l_{\tau\tau}}M^*_\nu\right)\ ,\\
\mathcal{F}^*,_s &= \frac{1}{a_\tau}M^*_\tau,_s - \frac{\gamma_{\tau\tau},_s}{a^3_\tau}M^*_\tau - \frac{1}{a_\tau}\sqrt{\frac{a}{\bar{a}}}\frac{1}{l_{\tau\tau}}\Bigg\{(l_{\nu\tau}m-\varphi_\tau m_\nu)M^*_\nu,_s + \Big[(l_{\nu\tau}m-\varphi_\tau m_\nu),_s\\
&\quad -(l_{\nu\tau}m-\varphi_\tau m_\nu)\Big(\frac{\gamma_{\tau\tau},_s}{a^2_\tau} + \frac{l_{\tau\tau},_s}{l_{\tau\tau}} + \frac{a}{\bar{a}}[(1+2\gamma_{\nu\nu})\gamma_{\tau\tau},_s + (1+2\gamma_{\tau\tau})\gamma_{\nu\nu},_s - 4\gamma_{\nu\tau}\gamma_{\nu\tau},_s]\Big)\Big]M^*_\nu\Bigg\}\ .
\end{aligned}
\tag{42}
$$

The boundary conditions (41) are again two-dimensionally exact for the shell reference surface, because no approximations are introduced into (41) besides those included in the postulated form (7) of the PVW. First such sets of four work-conjugate boundary conditions, associated with the function n_ν and the alternative polynomial definition of the surface bending tensor, was proposed by Pietraszkiewicz and Szwabowicz (1981).

Three static boundary conditions $(41)_{1-3}$ are also extremely complex partly as a result of decomposing $C^\alpha\bar{\mathbf{a}}_\alpha$, $\mathcal{D}\bar{\mathbf{n}}$, $\bar{b}^\alpha_\beta\tau^\beta\mathcal{F}^*\bar{\mathbf{a}}_\alpha$, and $\mathcal{F}^*,_s\bar{\mathbf{n}}$ in the undeformed boundary base $\{\boldsymbol{\nu},\boldsymbol{\tau},\mathbf{n}\}$. This is an unavoidable consequence of the fact that only the position vector $\mathbf{r}(s)$ of $\partial\mathscr{M}$ is assumed to be known in advance, and the position vector $\bar{\mathbf{r}}(s)$ of $\partial\bar{\mathscr{M}}$ is the one which should be found from the analysis. But the physical interpretation of the geometric boundary conditions in (41) is straightforward and all the fields in (41) are well defined in the known base $\{\boldsymbol{\nu},\boldsymbol{\tau},\mathbf{n}\}$.

5. Modified BVP and buckling shell problem in terms of displacements

To formulate the boundary value problem (BVP) in terms of displacements, the stress measures N_α^β and M_α^β should be eliminated from (21) and (41) by the constitutive equations, and then the strain measures γ_α^β and κ_α^β should be expressed through displacements using the strain-displacement relations (3). We briefly discuss below such BVPs for three simple cases of constitutive equations.

Within the first-approximation theory of thin shells made of homogeneous, isotropic, elastic material undergoing small strains, the strain energy density of the shell is given by

$$\begin{aligned} \Sigma &= \frac{h}{2} H^{\alpha\beta\lambda\mu} \left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu} \right) + O(Eh\eta^2\epsilon) \, , \\ H^{\alpha\beta\lambda\mu} &= \frac{E}{2(1+\nu)} \left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta}a^{\lambda\mu} \right) \, , \end{aligned} \tag{43}$$

and the corresponding constitutive equations are

$$\begin{aligned} N^{\alpha\beta} &= \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} = \frac{Eh}{1-\nu^2}\left[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma_\lambda^\lambda\right] + O\left(Eh\eta\epsilon\right) \, , \\ M^{\alpha\beta} &= \frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}} = \frac{Eh^3}{12(1-\nu^2)}\left[(1-\nu)\kappa^{\alpha\beta} + \nu a^{\alpha\beta}\kappa_\lambda^\lambda\right] + O\left(Eh^2\eta\epsilon\right) \, , \end{aligned} \tag{44}$$

where E and ν denote respectively Young's modulus and Poisson's ratio of the elastic material. Here η denotes the maximal strain in the shell space and ϵ describes formally the energetic error of this shell theory (Koiter, 1960; John, 1965). At any point $M \in \mathscr{M}$ the small parameter was described by Koiter (1960) to be $\sqrt{\epsilon} = \max\left(\frac{h}{b}, \frac{h}{l}, \frac{h}{L}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\right)$, where b is the distance of M from the lateral shell boundary, l - the smallest wavelength of geometric patterns of $\mathscr{M}$, L - the smallest wavelength of deformation patterns on $\mathscr{M}$, and R - the smallest radius of curvature of $\mathscr{M}$. If we substitute (44) into the BVP and reject the terms of the order of error introduced by the constitutive equations (44), then only the underlined terms in (21) and $(34)_{1,2}$ or (38) remain as the primary important terms.

Another example of the theory of shells based on the PVW (7) is the large-strain bending theory of elastic rubber-like shells. Various versions of such a theory were proposed for example by Chernykh (1980), Simmonds (1985), Schieck et al. (1992), and Libai and Simmonds (1998). In particular, when the greater eigenvalue γ of $\gamma_{\alpha\beta}$ was additionally assumed to be at most moderate, so that the approximation $1 + \gamma^2 \approx 1$ holds, the strain energy density was proposed by Pietraszkiewicz (2000) in possibly the simplest form

$$\Sigma = hW_{(0)}(\gamma_{\kappa\rho}) + \frac{h^3}{24} W_{(2)}^{\alpha\beta\lambda\mu}(\gamma_{\kappa\rho})[\kappa_{\alpha\beta}\kappa_{\lambda\mu}(1 - \gamma^{\sigma}_{\sigma})] \,, \tag{45}$$

where $W_{(0)}$ and $W_{(2)}^{\alpha\beta\lambda\mu}$ are the 3D strain energy density and its second derivative relative to $\gamma_{\kappa\rho}$, both taken at $\mathscr{M}$. Then the corresponding constitutive equations follow from

$$N^{\alpha\beta} = \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} \,, \quad M^{\alpha\beta} = \frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}} \,. \tag{46}$$

Using the estimate for moderate surface strains and other assumptions made while deriving (45), it is also possible to considerably simplify the equilibrium equations (21) and boundary conditions (41) by omitting many supposedly small terms of the same order, if necessary.

Still other examples of the theory of shells following from the PVW (7) are simple versions of the Lagrangian theory of elasto-plastic shells formulated entirely in terms of deformation of the reference surface, as discussed for example by Sawczuk (1980), Duszek (1982), and Schieck and Stumpf (1993). In those shell theories the equilibrium equations and boundary conditions corresponding to (21) and (41) can also be considerably simplified as well by omitting many supposedly small terms.

However, in the present paper we take a radically different approach. Accepting our inability to reasonably select a few critically important small terms in the shell equilibrium conditions among many other small terms which can be ignored, we do not simplify the shell relations at all in the process of elimination of the surface stress and strain measures. Due to enormous complexity of the resulting displacement shell relations such a BVP and the associated shell buckling problem cannot be derived just by hand transformations. In our approach this goal has been achieved with the help of some features provided by the symbolic language of Mathematica. Within this programming language two packages have been written: *ShellGeom.m* and *ShellBVP.m*.

The first package *ShellGeom.m* is responsible for generating all important characteristics of the assumed geometry of the shell reference surface $\mathscr{M}$ needed in transforming tensorial BVP to that expressed in partial derivatives. For the specified position vector $\mathbf{r} = \mathbf{r}(\theta^{\alpha})$ of the surface $\mathscr{M}$ this package is capable to generate analytic formulas for the local surface base on $\mathscr{M}$, its dual base, components of the first and second fundamental forms, Lame parameters, components of permutation tensor, Christoffel symbols as well as the mean and Gaussian curvatures. If the surface has a boundary contour $\partial\mathscr{M}$ parameterized by the arc-length parameter the package *ShellGeom.m* additionally generates several boundary characteristics such as the normal and geodesic curvatures and torsions of the boundary curve and of orthogonal to it surface curve as well as the components of boundary base vectors $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$.

The second package *ShellBVP.m* for the specified system of orthogonal coordinates $\{\theta^1, \theta^2\}$ derives the displacement equilibrium equations as well as the displacement natural static and geometric boundary conditions for the first–approximation geometrically non–linear theory of isotropic, elastic shells of the Kirchhof-Love type based on the formulation given in this paper. It also derives the corresponding shell buckling problem. In particular, this package performs the following steps:

- reads in the characteristics of the assumed shell geometry derived by the use of the package *ShellGeom.m*;
- introduces the constitutive equations (44) into (21) and (41) together with (38) and (42);
- expresses $\kappa_{\alpha\beta}$ in terms of $\chi_{\alpha\beta}$ defined in $(3)_2$;
- calculates covariant derivatives, performs summation over dummy indices and substitutes the geometrical characteristics of the reference surface;
- transforms the BVP to the equivalent non-dimensional form;
- multiplies the BVP by positive powers of $\sqrt{\frac{\bar{a}}{a}}$ in order to eliminate the square roots $\sqrt{\frac{\bar{a}}{a}}$ and then uses $(3)_4$;
- introduces into the non-dimensional BVP the strain-displacement relations transformed to the non-dimensional form;
- perturbates the BVP in displacements and derives the linearized homogeneous shell buckling equations together with corresponding work–conjugate sets of boundary conditions.

The package *ShellBVP.m* has some routines which check the correctness of the input arguments. Also the specified check points have been implemented into it to assure validity and correctness of derivation of the BVP. For example, one of the check points is the procedure which checks whether the compatibility conditions vanish identically after substitution into them the strain-displacement relations. If this is not true the main procedure interrupts derivation of the BVP. This significantly raises the confidence to the obtained BVP for each specified geometry of the shell. The output of this package is extremely large even for the relatively simple geometry of the cylinder, because no approximation is used during derivation of the BVP. Therefore, the resulting explicit displacement BVP and the corresponding shell buckling problem are available only in the computer memory and are not explicitly presented in this paper.

Let us briefly remind after Opoka and Pietraszkiewicz (2004) that the components of external loads $\mathbf{p}$, $\mathbf{c}$ and $\mathbf{N}^*$, $\mathbf{M}^*$ may be specified entirely independently, in general, by ten dimensionless parameters ρ_p forming the vector $\boldsymbol{\rho} \in \mathbf{R} \subset \mathbb{R}^{10}$. Then the non-linear BVP for a thin shell generated by the package *ShellBVP.m* from (21) and (41) with (44) and (3) can be presented symbolically as

$$f(\mathbf{u}; \boldsymbol{\rho}) = 0 \ , \tag{47}$$

where the non-linear continuosly differentiable operator f is defined on the product space $\mathscr{C}(\mathscr{M}, \mathbb{R}^3)$ with values in the Banach space, where $\mathscr{C}(\mathscr{M}, \mathbb{R}^3)$ is a set of all components of $\mathbf{u}$ and its gradients up to the 4th order. In engineering applications, however, all the external loads are usually specified by a single common parameter $\rho \in R \subset \mathbb{R}$, $\rho \geq 0$.

The solutions $\mathbf{u}_0(\rho)$ of (47), which can be reached starting form $\rho = 0$ in the undeformed state, form the primary equilibrium path. This path becomes unstable if an infinitesimally close adjacent equilibrium state $\mathbf{u}_1(\rho)$ exists for the same value of ρ.

In the neighbourhood of critical values of ρ we can replace $\mathbf{u}_1(\rho)$ by $\mathbf{u}_0(\rho) + \mathbf{u}$, where now $\mathbf{u}$ denotes the small increment of the displacements satisfying homogeneous boundary conditions. Substituting $\mathbf{u}_0(\rho) + \mathbf{u}$ into (47) we can linearize it with regard to $\mathbf{u}$ and take into account that $\mathbf{u}_0(\rho)$ should satisfy (47). As a result, we obtain the homogeneous linear shell buckling problem in terms of the incremental displacements $\mathbf{u}$ written again symbolically as

$$g(\mathbf{u}; \boldsymbol{\rho}) = 0 \ . \tag{48}$$

Non-trivial solutions of (48) can exist only at a discrete set of values of ρ, which are eigenvalues of the linear problem (48). The lowest positive eigenvalue $\rho_1 \equiv \rho_{crit}$ indicates the first bifurcation point at with the primary equilibrium path $\mathbf{u}_0(\rho)$ is intersected by a secondary equilibrium path $\mathbf{u}_1(\rho)$.

In the accompanying paper by Opoka and Pietraszkiewicz (2009, Appendix) we present in more detail the derivation of the BVP (47) and the corresponding buckling problem (48) for membrane prebuckling state in the special case of axially compressed circular cylinder. We also perform the extensive numerical analysis of bifurcation buckling for a wide range of length-to-diameter ratios of the cylinder under fourteen sets of work–conjugate boundary conditions.

6. Conclusions

We have formulated a new version of the Lagrangean non–linear theory of thin shells expressed in terms of displacements of the shell reference surface as the only independent field variables. The formulation has been based on the principle of virtual work postulated for the reference surface. Both the equilibrium equations and the set of four work–conjugate static and geometric boundary conditions are derived exactly from the PVW without using any kind of approximations. Elimination of the surface stress and strain measures in terms of displacements is performed exactly as well without using the approximate nature of the constitutive equations to simplify the BVP. The latter steps have been performed automatically with the help of two packages set up in MATHEMATICA. The final BVP and the corresponding shell buckling problem are obviously extremely complex. They are manageable only as the relations given in the computer memory, not as those explicitly written on the paper. By taking into account all supposedly small terms in the buckling shell equations we are sure that among them are also those a few supposedly small terms which may appear to be critically important ones in finding the correct buckling load of thin shells sensitive to imperfections.

The idea of the present report has grown from our experience gained in the paper Opoka and Pietraszkiewicz (2009) while performing the refined analysis of bifurcation buckling for the axially compressed circular cylinder, which is one of the most imperfection–sensitive structural problems known in the literature. We advise the reader to consult this accompanying paper to better understand the reasons why the present paper has been written.

Currently the package *ShellBVP.m* generates the two-dimensionally exact displacement BVP and the shell buckling problem only with the constitutive equations (44) valid for the first-approximation geometrically non–linear theory of thin, isotropic elastic shells. Exactly the same approach can also be used with any of the constitutive equations of large-strain theory of rubber-like shells. But this type of approach may also be useful in formulating any two-dimensionally exact version of the Lagrangian non–linear theory of elasto-plastic shells as well, if peculiar character of performing the analysis in the plastic range is taken into account.

References

Budiansky, B., 1968. Notes on non-linear shell theory. Journal of Applied Mechanics 35 (2), 393–401.

Chernykh, K. F., 1964. Linear Theory of Shells. Vol. 2. Leningrad St. Univ. Press, (in Russian), English transl. : NASA-TT-F-II 562, 1968.

Chernykh, K. F., 1980. Nonlinear theory of isotropically elastic thin shells. Mechanics of Solids 15, 118–127, transl. of Izv. AN SSSR, MTT.

Chien, W. Z., 1944. The intrinsic theory of thin shells and plates. Part I - General theory. Quarterly of Applied Mathematics 1 (4), 297 – 327.

Ciarlet, P. G., 2005. Introduction to Differential Geometry with Applications to Elasticity. Springer, Heidelberg.

Danielson, D. A., 1970. Simplified intrinsic theory for arbitrary elastic shells. International Journal of Engineering Sciences 8 (3), 251–259.

Duszek, M., 1982. Foundations of the Non-linear Plastic Shell Theory. Ruhr-Universität, Inst. für Mech. , Mitt. 31, Bochum.

Flügge, W., 1972. Tensor Analysis and Continuum Mechanics. Springer, Berlin.

Galimov, K. Z., 1951. On the general theory of plates and shells with finite displacements and deformations (in Russian). Prikladnaya Matematika i Mekhanika 15 (6), 723–742.

Green, A. E., Zerna, W., 1968. Theoretical Elasticity, 2nd ed. Clarendon Press.

John, F., 1965. Estimates for the derivatives of the stresses in a thin shell and interior shell equations. Communications on Pure and Applied Mathematics 18 (2), 235–267.

Koiter, W. T., 1960. A cosistent first approximation in the general theory of shells. In: The Theory of Thin Elastic Shells. Proceedings of the IUTAM Symposium, Delft, 1959. North Holland, Amsterdam, pp. 12–33.

Koiter, W. T., 1966. On the nonlinear theory of thin elastic shells. In: Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen. Vol. 69 of Series B. Amsterdam, pp. 1–54.

Koiter, W. T., Simmonds, J. G., 1973. Foundations of shell theory. In: Becker, E., Mikhailov, G. K. (Eds.), Proceedings of the Thirteenth International Congress on Theoretical and Applied Mechanics, Moscow, 1972. Springer, Berlin, pp. 150–176.

Libai, A., Simmonds, J. G., 1998. The Nonlinear Theory of Elastic Shells, 2nd Edition. Cambridge University Press, Cambridge.

Makowski, J., Pietraszkiewicz, W., 1989. Work-conjugate boundary conditions in the nonlinear theory of thin shells. ASME Journal of Applied Mechanics 56 (2), 395–402.

Nolte, L. P., 1983. Contribution to the derivation and comparative investigation of geometrically non-linear shell theories undergoing large rotations (in German). Ruhr-Universität, Inst. für Mech. , Mitt. 39, Bochum.

Nolte, L. P., Makowski, J., Stumpf, H., 1986. On the derivation and comparative analysis of large rotation shell theories. Ingenieur-Archiv 56 (2), 145–160.

Novozhilov, V. V., Shamina, V. A., 1975. On kinematic boundary conditions in nonlinear problems of theory of elasticity (In Russian). Izvestiya AN SSSR, Mekhanika Tverdogo Tela 10 (5), 63–71.

Opoka, S., Pietraszkiewicz, W., 2004. Intrinsic equations for non-linear deformation and stability of thin elastic shells. International Journal of Solids and Structures 41, 3275–3292.

Opoka, S., Pietraszkiewicz, W., 2009. On refined analysis of bifurcation buckling for the axially compressed circular cylinder (submitted to International Journal of Solids and Structures).

Pietraszkiewicz, W., 1974. Lagrangian nonlinear theory of shells. Archives of Mechanics 26 (2), 221–228.

Pietraszkiewicz, W., 1977. Introduction to the Nonlinear Theory of Shells. Ruhr-Universität, Inst. für Mech. , Mitt. 10, Bochum.

Pietraszkiewicz, W., 1980. Finite rotations in the non-linear theory of thin shells. In: Olszak, W. (Ed.), Thin Shell Theory: New Trends and Appliactions. Springer, Wien, pp. 153–208.

Pietraszkiewicz, W., 1984. Lagrangian description and incremental formulation in the non-linear theory of thin shells. International Journal of Non-Linear Mechanics 19 (2), 115–140.

Pietraszkiewicz, W., 1989. Geometrically nonlinear theories of thin elastic shells. Advances in Mechanics 12 (1), 51–130.

Pietraszkiewicz, W., 1993. Work-conjugate boundary conditions associated with the total rotation angle of the shell boundary. ASME Journal of Applied Mechanics 60 (3), 785–786.

Pietraszkiewicz, W., 2000. On the Alumäe type non-linear theory of thin irregular shells. In: Izvestiia VUZov, Severo-Kavk. Region, Estestvennye nauki, Spetsvypusk. pp. 127–136.

Pietraszkiewicz, W., Szwabowicz, M. L., 1981. Entirely Lagrangian nonlinear theory of thin shells. Archives of Mechanics 33 (2), 273–288.

Sanders, J. L., 1963. Nonlinear theories for thin shells. Quarterly of Applied Mathematics 21 (1), 21–36.

Sawczuk, A., 1980. On plastic analysis of shells. In: Koiter, W. T., Mikhailov, G. K. (Eds.), Theory of Shells. North-Holland, Amsterdam, pp. 27–63.

Schieck, B., Pietraszkiewicz, W., Stumpf, H., 1992. Theory and numerical analysis of shells undergoing large elastic strains. International Journalof Solids and Structures 29 (6), 689–709.

Schieck, B., Stumpf, H., 1993. Deformation analysis for finite elasto-plastic strains in a lagrangean-type description. International Journal of Solids and Structures 30 (19), 2639–2660.

Schmidt, R., Pietraszkiewicz, W., 1981. Variational principles in the geometrically non-linear theory of shells undergoing moderate rotations. Ingenieur-Archiv 50, 187–201.

Simmonds, J. G., 1985. The strain energy density of rubber-like shells. International Journal of Solids and Structures 21, 67–77.

Tovstik, P., Smirnov, A. L., 2001. Asymptotic Methods in the Buckling Theory of Elastic Shells. World Scientific, Singapore.