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On tension of a two-phase elastic tube

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ABSTRACT: Within the thermomechanics of the Cosserat-type shells undergoing the diffusionless (displacive) phase transitions developed by Eremeyev & Pietraszkiewicz (2009), we propose the thermodynamic condition allowing one to determine quasistatic motion of the phase interface on the deformed shell base surface. The theoretical model is illustrated by example of a thin-walled circular cylindrical tube made of a two-phase elastic material subject to tensile forces on the one end and clamped on another one. The solution reveals existence of the hysteresis loop which size depends upon values of several loading parameters.

1 INTRODUCTION

Phase transition (PT) phenomenon in continuous media originally described by Gibbs in 1875–1878, see Gibbs (1928), was developed in a number of papers summarised in several recent books for example by Bhattacharya (2003), Abeyaratne & Knowles (2006), Lagoudas (2008), and Berezovski et al. (2008). In this approach one assumes existence of the sharp phase interface being a sufficiently regular surface dividing different material phases. The position and motion of the phase interface itself is among the most discussed issues in the field. In the literature many model one-dimensional (1D) problems were analysed theoretically, numerically and experimentally which adequately described behaviour of bars, rods, and beams made of martensitic materials.

However, experiments on shape memory alloys and other materials undergoing PT are often performed with thin-walled samples such as thin strips, rectangular plates or thin tubes.

The non-linear equilibrium conditions of elastic shells undergoing PT of martensitic type were formulated by Eremeyev & Pietraszkiewicz (2004) and Pietraszkiewicz et al. (2007) within the dynamically and kinematically exact theory of shells developed by Libai & simmonds (1998), Chróścielewski et al. (2004), and Eremeyev & Zubov (2008). In this shell theory the translation vector $\boldsymbol{u}$ and rotation tensor $\boldsymbol{Q}$ fields are the only independent variables. By analogy to the 3D case, the two-phase shell was regarded as the Cosserat surface consisting of two material phases divided by a sufficiently smooth surface curve. Existence of such a curve was confirmed by several experiments on thin-walled samples.

2 BASIC RELATIONS OF SHELL THERMOMECHANICS

In the undeformed placement the shell is represented by the base surface M described by the position vector $\boldsymbol{x}(\theta^\alpha)$, and orientation of M is defined by the unit normal vector $\boldsymbol{n}(\theta^\alpha)$, with $\{\theta^\alpha\}$, $\alpha = 1, 2$ the surface curvilinear coordinates.

In the deformed placement the shell is represented by the position vector $\boldsymbol{y} = \chi(\boldsymbol{x})$ of the deformed material base surface $N = \chi(M)$ with attached three directors $(\boldsymbol{d}_\alpha, \boldsymbol{d})$ such that

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}, \quad \boldsymbol{d}_\alpha = \boldsymbol{Q}\boldsymbol{x}_{,\alpha}, \quad \boldsymbol{d} = \boldsymbol{Q}\boldsymbol{n}, \tag{1}$$

where χ is the deformation function, $\boldsymbol{u} \in E$ the translation vector of M, and $\boldsymbol{Q} \in SO(3)$ the proper orthogonal tensor, $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$, $\det \boldsymbol{Q} = +1$, representing the work-averaged gross rotation of the shell cross sections from their undeformed shapes described by $(\boldsymbol{x}_{,\alpha}, \boldsymbol{n})$.

In the shell undergoing phase transition above some level of deformation it is assumed that different material phases A and B may appear in different complementary subregions N_A and N_B separated by the curvilinear phase interface $\mathcal{D} \in N$. For a piecewise differentiable mapping χ we can introduce on M a singular image curve $\mathcal{C} = \chi^{-1}(\mathcal{D})$ separating the corresponding image regions $M_A = \chi^{-1}(N_A)$ and $M_B = \chi^{-1}(N_B)$.

The two-dimensional (2D) local laws of shell thermomechanics can be derived by direct and exact through-the-thickness integration of global 3D balances of forces, moments, energy and the entropy inequality, see Eremeyev & Pietraszkiewicz (2009).

After appropriate transformations the resulting 2D local Lagrangian laws in $M \backslash C$ become

$$
\begin{aligned}
&Div_s \boldsymbol{N} + \boldsymbol{f} = \boldsymbol{0}, \\
&Div_s \boldsymbol{M} + ax(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T) + \boldsymbol{c} = \boldsymbol{0}, \\
&\rho \tfrac{d\varepsilon}{dt} = \rho(q^+ + q^- + q_\Pi) - Div_s \boldsymbol{q} \\
&\qquad + \boldsymbol{N} \bullet \boldsymbol{E}^\circ + \boldsymbol{M} \bullet \boldsymbol{K}^\circ, \\
&\rho \tfrac{d\psi}{dt} \le \rho\eta \tfrac{dT}{dt} + \boldsymbol{N} \bullet \boldsymbol{E}^\circ + \boldsymbol{M} \bullet \boldsymbol{K}^\circ + Grad_s\left(\tfrac{1}{T}\right) \cdot \boldsymbol{q} \\
&\qquad + \rho q^+ \left(1 - \tfrac{T}{T^+_{ext}}\right) + \rho q^- \left(1 - \tfrac{T}{T^-_{ext}}\right),
\end{aligned}
\tag{2}
$$

where $\boldsymbol{f}$, $\boldsymbol{c}$ are the resultant surface force and couple vector fields acting on $N \backslash \mathcal{D}$, but measured per unit area of $M \backslash C$, $(\boldsymbol{N}, \boldsymbol{M}) \in E \otimes T_x M$ the surface stress resultant and stress couple tensors of the first Piola-Kirchhoff type, $\boldsymbol{F} = Grad_s \boldsymbol{y}$ the surface deformation gradient, $\boldsymbol{F} \in E \otimes T_x M$, $ax(\dots)$ the axial vector associated with the skew tensor $(\dots)$, $(\boldsymbol{E}^\circ, \boldsymbol{K}^\circ) \in E \otimes T_x M$ the corotational variations of the shell strain measures work-conjugate to $(\boldsymbol{N}, \boldsymbol{M})$, and Div_s the surface divergence operator on M. Additionally, ε and η are the surface internal energy and entropy densities, ρ the undeformed surface mass density, $q^\pm$ the heat influx densities through the upper $(+)$ and lower $(-)$ shell faces, q_Π the internal surface heat supply density, $\boldsymbol{q}$ the surface heat influx vector, T the through-the-thickness average temperature, T^+_{ext} and T^-_{ext} temperatures of the external media surrounding the shell from above and below, and $\psi = \varepsilon - T\eta$ the surface free energy density.

Along the curvilinear phase interface C, which is the quasistatically moving singular curve on M, after appropriate transformations we also obtain the local Lagrangian jump conditions

$$
[\![\boldsymbol{N}\boldsymbol{\nu}]\!] = 0, \quad [\![\boldsymbol{M}\boldsymbol{\nu}]\!] + [\![\boldsymbol{y} \times \boldsymbol{N}\boldsymbol{\nu}]\!] = 0, \tag{3}
$$

$$
V[\![\rho\varepsilon]\!] + [\![\boldsymbol{N}\boldsymbol{\nu} \cdot \boldsymbol{v}]\!] + [\![\boldsymbol{M}\boldsymbol{\nu} \cdot \boldsymbol{\omega}]\!] - [\![\boldsymbol{q} \cdot \boldsymbol{\nu}]\!] = 0. \tag{4}
$$

$$
V[\![\rho\eta]\!] - [\![\tfrac{1}{T}\boldsymbol{q} \cdot \boldsymbol{\nu}]\!] \equiv \delta^2 \ge 0. \tag{5}
$$

where the expression $[\![\dots]\!] = (\dots)_B - (\dots)_A$ means the jump at C, $\boldsymbol{\nu}$ the surface unit vector externally normal to ∂M, and δ^2 represents creation of entropy at the interface C.

For the coherent phase interface both fields $\boldsymbol{y}$ and $\boldsymbol{Q}$ are supposed to be continuous at C and the kinematic compatibility conditions along C become

$$
[\![\boldsymbol{v}]\!] + V[\![\boldsymbol{F}\boldsymbol{\nu}]\!] = 0, \quad [\![\boldsymbol{\omega}]\!] + V[\![\boldsymbol{K}\boldsymbol{\nu}]\!] = 0, \tag{6}
$$

where $\boldsymbol{v} = \dot{\boldsymbol{u}}$ is the virtual translational vector, $\boldsymbol{\omega} = ax(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T)$ the virtual rotation vector, $V = \dot{\boldsymbol{x}}_C \cdot \boldsymbol{\nu}$ the exterior normal virtual translation of the phase curve C, and t a time-like scalar parameter.

For the coherent phase interface

$$
T\delta^2 = -V\left\{[\![\rho\psi]\!] - \boldsymbol{\nu} \cdot \boldsymbol{N}^T[\![\boldsymbol{F}\boldsymbol{\nu}]\!] - \boldsymbol{\nu} \cdot \boldsymbol{M}^T[\![\boldsymbol{K}\boldsymbol{\nu}]\!]\right\} \quad \text{at } C.
$$

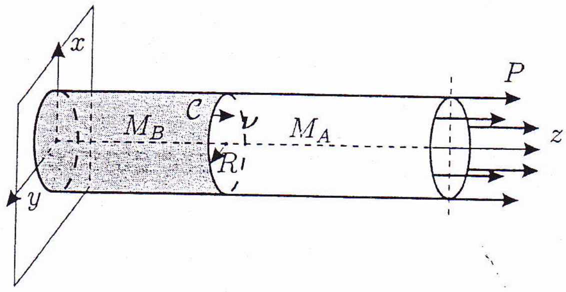

Figure 1. Tension of the thin-walled two-phase tube.

The entropy production δ^2 remains always non-negative for all thermodynamic processes. This allows us to postulate the kinetic equation, describing motion of the phase interface for all quasistatic processes, in the form

$$
V = -\mathcal{F}(\boldsymbol{\nu} \cdot [\![\boldsymbol{C}]\!]\boldsymbol{\nu}), \quad \boldsymbol{C} = \rho\psi\mathbf{A} - \boldsymbol{N}^T\boldsymbol{F} - \boldsymbol{M}^T\boldsymbol{K}, \tag{7}
$$

where $\mathcal{F}$ is the non-negative definite kinetic function depending on the jump of $\boldsymbol{C}$ at C, i.e. $\mathcal{F}(\varsigma) \ge 0$ for $\varsigma > 0$, and $\mathbf{A} = \mathbf{1} - \boldsymbol{n} \otimes \boldsymbol{n}$.

After Berezovski et al. (2008), we assume $\mathcal{F}(\varsigma)$ in the form

$$
\mathcal{F}(\varsigma) = \begin{cases} \dfrac{k(\varsigma - \varsigma_0)}{1 + a(\varsigma - \varsigma_0)} & \varsigma \ge \varsigma_0, \\ 0 & -\varsigma_0 < \varsigma < \varsigma_0, \\ \dfrac{k(\varsigma + \varsigma_0)}{1 - a(\varsigma + \varsigma_0)} & \varsigma \le -\varsigma_0. \end{cases} \tag{8}
$$

Here ς_0 describes the effects associated with nucleation of the new phase and action of the surface tension, a is a parameter describing the limit value of the phase transition virtual translation, and k is a positive kinetic factor.

Summarising, the BVP for the shell undergoing phase transitions consists of the equilibrium equations $(2)_{1,2}$ supplemented by appropriate static and kinematic boundary conditions for $\boldsymbol{u}$ and $\boldsymbol{Q}$, the energy transfer equation $(2)_3$ with appropriate boundary conditions for T, the surface entropy inequality $(2)_4$, as well as the balance equations (3), (6), and (7) along the interface C. The equation $(7)_1$ is used to find position of the curvilinear interface C in its quasistatic motion.

3 EXAMPLE: TENSION OF TWO-PHASE TUBE

We discuss the thin circular cylindrical shell of length L, radius R, and thickness h made of material undergoing phase transition. The tube is extended by forces P uniformly distributed at the right shell boundary, Figure 1. The left shell boundary at $z = 0$ is clamped. We assume that the shell deformation is infinitesimal. We also assume that the deformation process is isothermic, and additionally that $T = T^+_{ext} = T^-_{ext} = \text{const}$ and $q_\Pi = 0$. In such a case the problem is reduced to the stress-induced phase transition.

We consider the 2D polar-elastic strain energy densities of the isotropic phases, see Eremeyev

& Pietraszkiewicz (2009), with phase transformation stretch and bending measures. The both material phases differ by values of the elastic moduli as well as by values of the energy densities in the undeformed state.

Under condition given above there exists axisymmetric deformation state

$$\boldsymbol{u} = u(z)\boldsymbol{e}_z + w(z)\boldsymbol{e}_r, \quad \boldsymbol{\varphi} = \varphi(z)\boldsymbol{e}_\varphi. \tag{9}$$

The discussed example can be reduced to solving the boundary-value problem consisting of the following system of ordinary differential equations:

$$\begin{aligned}
&N'_{zz} = 0, \quad N'_{rz} = \tfrac{N_{\phi\phi}}{R}, \quad M'_{\phi z} = -\tfrac{M_{r\phi}}{R} - N_{rz},\\
&N_{zz} = C(u' - \epsilon_p) + C\nu(w/R - \epsilon_p),\\
&N_{rz} = \alpha_s C(1-\nu)(w' - \varphi),\\
&N_{\phi\phi} = C\nu(u' - \epsilon_p) + C(w/R - \epsilon_p),\\
&M_{\phi z} = D(1-\nu)\varphi', \quad M_{r\phi} = -\alpha_t D(1-\nu)\tfrac{\varphi}{R},\\
&u(0) = w(0) = \varphi(0) = 0,\\
&N_{zz}(L) = P, \quad N_{rz}(L) = M_{\phi z}(L) = 0,
\end{aligned} \tag{10}$$

where C, D, ν, α_s, and α_t are elastic moduli, while ϵ_p is the phase transformation strain. This is the system of ODE with constant coefficients expressed in terms of independent functions u, w, φ. The system (10) has always the particular solution

$$\begin{aligned}
&u(z) = u_p(z) \equiv \left(\tfrac{P}{C(1-\nu^2)} + \epsilon_p\right) z + \text{const},\\
&w(z) = w_p \equiv -\left(\tfrac{P\nu}{C(1-\nu^2)} - \epsilon_p\right) R, \quad \varphi = 0,
\end{aligned} \tag{11}$$

for which $N_{zz} = P$, $N_{rz} = 0$, $N_{\phi\phi} = 0$, $\boldsymbol{M} = \boldsymbol{0}$. This solution describes the axisymmetric membrane equilibrium state of the cylinder. In the two-phase cylinder such a solution is possible only when $\nu_A = \nu_B = 0$ or $\frac{\nu_A}{C_A(1-\nu_A^2)} = \frac{\nu_B}{C_B(1-\nu_B^2)}$ and $\epsilon_p^A = \epsilon_p^B$, since otherwise, according to (11), normal translations of parts A and B would not coincide: $w_A \neq w_B$.

We first solve the simplest case when $\nu_A = \nu_B = 0$. This problem becomes entirely analogous to the 1D problem discussed by Abeyaratne & Knowles (2006) as a model problem of the 3D continuum model of PT. The relation how the force P depends on deformation in the equilibrium states is illustrated in Figure 2. Here $E_L = u'(L)$.

If the quasistatic motion of C is governed by the kinetic equation (7), in Figure 2 the respective graphs AB' describe the loading and BA' the unloading. As a result, in the deformation process we observe the existence of the hysteresis loop $AB'BA'$ characteristic to PT of martensitic type. The size of the loop depends essentially on the form of function $\mathcal{F}$, and particularly upon values of the kinetic factor k and the parameter P_0 determining the loading velocity. When $\hat{k} \equiv k/P_0$ increases the area of hysteresis loop decreases. Examples of several deformation paths for different values of $\hat{k}$ are given in Figure 2. It is seen that with the growing $\hat{k}$ we obtain the narrowing loops $AB'BA'$, $AB''BA''$,

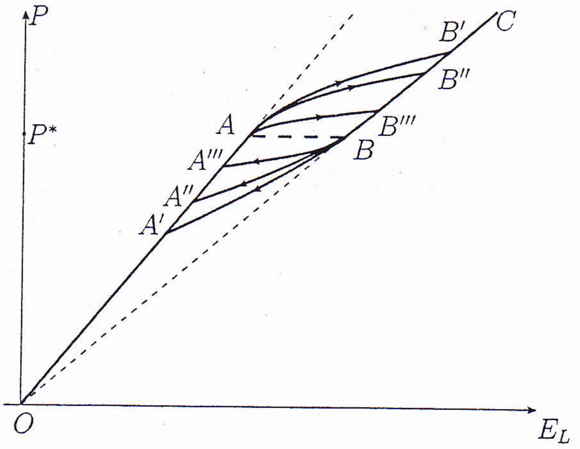

Figure 2. $P - E_L$ curves for two-phase shell for different values of $\hat{k}$.

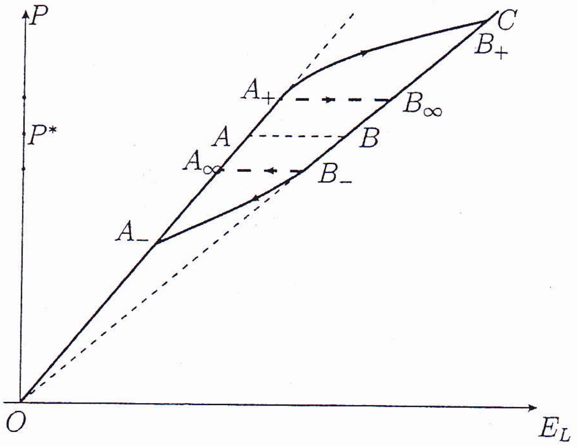

Figure 3. $P - E_L$ curves for $\varsigma_0 \neq 0$.

$AB'''BA'''$, etc. The limit $\hat{k} \to \infty$ corresponds both to the infinitely large kinetic factor $k \to \infty$ and to the infinitely small loading velocity $P_0 \to 0$. In the limit $\hat{k} \to \infty$ the hysteresis loop reduces to the equilibrium segment AB when $P = P^*$. This means in particular that with the infinitely small loading velocity the deformation follows the equilibrium path $OABC$.

When $\varsigma_0 \neq 0$, the corresponding relation $P(E_L)$ is shown in Figure 3. In this case the size of hysteresis loop becomes larger with the growing value of ς_0. But for $\hat{k} \to \infty$ the limiting paths reduce to two different respective segments A_+B_∞ and B_-A_∞, and the hysteresis loop takes place also in this limit case.

In the general case the solutions of (10) for translation and rotations is more complicated. In particular, for w we obtain

$$\begin{aligned}
&w = w_\circ(z) + w_p,\\
&w_\circ = e^{-\omega_R \bar{z}}(c_1 \cos\omega_I \bar{z} + c_2 \sin\omega_I \bar{z})\\
&\quad + e^{\omega_R \bar{z}}(c_3 \cos\omega_I \bar{z} + c_4 \sin\omega_I \bar{z}), \qquad \bar{z} = z/R,
\end{aligned} \tag{12}$$

where c_k, $k = 1, ..., 4$, are integration constants,

$$\{\omega_R, \omega_I\} = \{\text{Re}, \text{Im}\}\sqrt{2\eta_1 + 2i\sqrt{4\eta_2 - {\eta_1}^2}/2},$$

$$\eta_1 = \tfrac{\alpha_s\alpha_t + 1 + \nu}{\alpha_s}, \quad \eta_2 = (1+\nu)\left(12\delta^{-2} + \tfrac{\alpha_t}{\alpha_s}\right),$$

$$\varsigma = (1+\nu)\left(12\delta^{-3} + \alpha_t\delta^{-1}\right), \quad \delta = h/R.$$

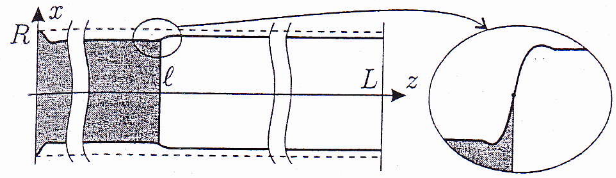

Figure 4. Shape of the thin-walled two-phase tube after phase transition (magnified).

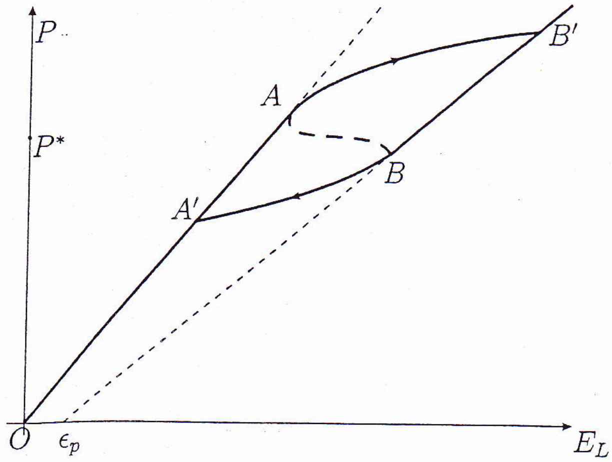

Figure 5. $P-E_L$ curves following from the general solution.

For a thin tube $\eta_2 \gg \eta_1$. Indeed, if one takes $\nu = 1/3$, $\alpha_s = 5/6$, $\alpha_t = 7/10$ and $\delta = 0.1$, then $\eta_1 = 2.3$ and $\eta_2 = 1601.12$, $\omega_R = 4.537$ and $\omega_I = 4.408$, respectively. Hence, we can apply some asymptotic formulae for the boundary layers.

The general solution of (10) differs essentially from the previous membrane one, because now we have also the boundary layer solutions in the neighbourhood of the clamped edge and the phase interface, see Figure 4. The boundary layer parts of the solutions quickly decay, and far from the clamping and the phase boundary w becomes constant coinciding with w_p.

Dependence of w upon z leads to the qualitative and quantitative differences of the general solution as compared with the membrane solution discussed in Figures 2 and 3. In particular, the equilibrium part of the diagram $P - E_L$ now becomes not a horizontal segment as before, see AB in Figure 5. The influence of boundary layer parts of the solution manifests itself most when $\ell \sim 0$ and $\ell \sim L$, i.e. at the shell edges. Also the shape and size of the hysteresis loop becomes different when the general solution is used.

The proposed 2D model allows one to take into account several additional factors unavailable in the existing 1D models of phase transitions, such as solutions of the boundary layer type or more differentiated ways of loading and unloading. We are also able to analyse even analytically quite complex problems which in the 3D models are possible to discuss only by numerical methods.

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