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# On exact two-dimensional kinematics for the branching shells

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ABSTRACT: We construct the two-dimensional (2D) kinematics which is work-conjugate to the exact 2D local equilibrium conditions of the non-linear theory of branching shells. It is shown that the compatible shell displacements consist of the translation vector and rotation tensor fields defined on the regular parts of the shell base surface as well as independently on the singular surface curve modelling the shell branching. Several characteristic types of the junctions are discussed and for each of them the explicit form of the principle of virtual work is suggested.

## 1 INTRODUCTION

Konopińska & Pietraszkiewicz (2007) and Konopińska (2007) formulated the exact, two-dimensional (2D) equilibrium conditions (3)–(6) for the non-linear theory of branching and self-intersecting shells. The conditions are derived by performing direct through-the-thicknes integration in the global 3D equilibrium conditions of continuum mechanics. The results do not depend on the value of shell thickness, the internal through-the-thickness shell structure, material properties, and are valid for an arbitrary deformation of the shell material elements.

In this note we construct a dual mathematical structure representing an exact 2D kinematics on the irregular shell base surface $M$ for the branching shell. The kinematics is work conjugate to the exact 2D equilibrium conditions (3)–(6). We begin with the integral identity (7) and then interpret some arbitrary vector fields as the kinematically admissible virtual displacements corresponding to the real shell deformation. This allows us to construct the 2D exact expression (12) for the principle of virtual work (PVW) defined on the shell base surface $M$ having the singular surface curve $\Gamma$ modelling the shell branching. As a result, the shell displacements are expressed by the work-averaged translation vector $\boldsymbol{u}$ and rotation tensor $\boldsymbol{Q}$ fields describing the gross deformation of the shell cross section independently in $M \setminus \Gamma$ and along $\Gamma$. Discussing relations between limits at $\Gamma$ of the displacements defined in $M \setminus \Gamma$ and of those defined only along $\Gamma$, we are able to characterise several types of shell junctions at the singular curve $\Gamma$.

## 2 NOTATION AND LOCAL EQUILIBRIUM CONDITIONS

A shell is a 3D thin solid body identified in a reference (undeformed) placement with a region B of the physical space $\mathcal{E}$ having $E$ as its 3D translation vector space. The position vector $\mathbf{x}$ of any point $\mathrm{x} \in \mathrm{B}$ can be given by

$$\mathbf{x}(x,\xi) = \boldsymbol{x}(x) + \xi \boldsymbol{t}(x) \,, \tag{1}$$

where $\boldsymbol{x}(x) = \mathbf{x}(x,0)$ is the position vector of a point $x$ of some base surface $M$, while $\xi$ is the distance to x along the unit vector $\boldsymbol{t}$ not necessarily normal to $M$.

The position vector $\mathbf{y} = \chi(\mathbf{x})$ of any shell point y in the deformed placement $\bar{\mathrm{B}} = \chi(\mathrm{B})$ can always be represented by

$$\mathbf{y}(x,\xi) = \boldsymbol{y}(x) + \boldsymbol{\zeta}(x,\xi) \,, \quad \boldsymbol{\zeta}(x,0) = \mathbf{0} \,, \tag{2}$$

where $\boldsymbol{y} = \chi(\boldsymbol{x})$ is the position vector of the deformed material base surface $\overline{M} = \chi(M)$, and $\boldsymbol{\zeta}$ is a deviation of $\mathrm{y} \in \mathrm{B}$ from $\overline{M} = \chi(M)$.

For the branching shell Konopińska & Pietraszkiewicz (2007) worked out the through-the-thickness integration procedure leading to the *exact* 2D local equilibrium conditions for any part $\Pi \in M$ having the singular surface curve $\Gamma$ modelling the common junction of several regular branches $M_k, k = 1, \ldots, n$, of $M$. In the referential description these local equilibrium conditions consist of: the equilibrium equations in $\Pi \subset M \setminus \Gamma$

$$\mathrm{Div}_s \boldsymbol{N} + \boldsymbol{f} \equiv \widetilde{\boldsymbol{f}} = \boldsymbol{0} ,$$
$$\mathrm{Div}_s \boldsymbol{M} + \mathrm{ax}\left(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T\right) + \boldsymbol{c} \equiv \widetilde{\boldsymbol{c}} = \boldsymbol{0} ; \tag{3}$$

the static boundary conditions along $\partial\Pi_f \subset \partial M_f$

$$\boldsymbol{n}^* - \boldsymbol{N}\nu \equiv \widetilde{\boldsymbol{n}} = \boldsymbol{0} , \quad \boldsymbol{m}^* - \boldsymbol{M}\nu \equiv \widetilde{\boldsymbol{m}} = \boldsymbol{0} ; \tag{4}$$

the static continuity conditions along $\Gamma \cap \Pi$

$$\boldsymbol{n}' + [\boldsymbol{N}\nu] + \boldsymbol{f}_\Gamma \equiv \widetilde{\boldsymbol{f}_\Gamma} = \boldsymbol{0} ,$$
$$\boldsymbol{m}' + \boldsymbol{y}'_\Gamma \times \boldsymbol{n} + [\boldsymbol{M}\nu] + \boldsymbol{c}_\Gamma \equiv \widetilde{\boldsymbol{c}_\Gamma} = \boldsymbol{0} ; \tag{5}$$

and the static boundary conditions

$$\boldsymbol{n}_i^* - \boldsymbol{n}_i \equiv \widetilde{\boldsymbol{n}_i} = \boldsymbol{0} , \quad \boldsymbol{m}_i^* - \boldsymbol{m}_i \equiv \widetilde{\boldsymbol{m}_i} = \boldsymbol{0} ,$$
$$\boldsymbol{n}_e^* - \boldsymbol{n}_e \equiv \widetilde{\boldsymbol{n}_e} = \boldsymbol{0} , \quad \boldsymbol{m}_e^* - \boldsymbol{m}_e \equiv \widetilde{\boldsymbol{m}_e} = \boldsymbol{0} \tag{6}$$

at the singular points $x_i, x_e \in \Gamma \cap \partial M_f$.

In (3)–(6), $(\boldsymbol{N}, \boldsymbol{M}) \in E \otimes T_x M$ are the surface stress resultant and stress couple tensors of the 1st Piola-Kirchhoff type, $(\boldsymbol{f}, \boldsymbol{c}) \in E$ the surface resultant force and couple vectors, $\mathrm{Grad}_s$ and $\mathrm{Div}_s$ the surface gradient and divergence operators on $M$, $(\boldsymbol{n}^*, \boldsymbol{m}^*) \in E$ the boundary resultant force and couple vectors along $\partial M_f$, $(\boldsymbol{f}_\Gamma, \boldsymbol{c}_\Gamma) \in E$ the curvilinear resultant force and couple vectors along $\Gamma$, $(\boldsymbol{n}, \boldsymbol{m}) \in E$ the curvilinear vectors generated along $\Gamma$ by the concentrated vectors $\boldsymbol{n}_i^*$, $\boldsymbol{m}_i^*$ and $\boldsymbol{n}_e^*$, $\boldsymbol{m}_e^*$ applied at the initial $x_i$ and end $x_e$ points of $\Gamma$, respectively. Additionally, $\mathrm{ax}(\boldsymbol{A})$ means the axial vector of the skew tensor $\boldsymbol{A}$, $\nu \in T_x M$ the unit vector externally normal to $\partial\Pi$, $[\boldsymbol{a}]$ the jump of the vector field $\boldsymbol{a}(x)$ at the singular surface curve $\Gamma$, and $(.)' \equiv \frac{\mathrm{d}}{\mathrm{d}s}(.)$.

The relations (3) and (4) are equivallent to those given for the regular shell for example by Labai & Simmonds (1983) and Makowski & Stumpf (1990). The static relations (5) and (6) complete by some correcting terms various analogous approximate relations proposed by Mokowski & Stumpf (1994), Chróścielewski et al. (1997, 2004) and Pietraszkiewicz (2001) using alternative approximate reduction procedures.

## 3 WORK-CONJUGATE SHELL KINEMATICS

Let $(\boldsymbol{v}, \boldsymbol{w}) \in E$ and $(\boldsymbol{v}_\Gamma, \boldsymbol{w}_\Gamma) \in E$ be two pairs of smooth vector fields on $M \setminus \Gamma$ and $\Gamma$, respectively. Then we can set the integral identity

$$\iint_{M\setminus\Gamma} \left(\widetilde{\boldsymbol{f}} \cdot \boldsymbol{v} + \widetilde{\boldsymbol{c}} \cdot \boldsymbol{w}\right) \mathrm{d}a + \int_{\partial M_f} (\widetilde{\boldsymbol{n}} \cdot \boldsymbol{v} + \widetilde{\boldsymbol{m}} \cdot \boldsymbol{w})\, \mathrm{d}s$$
$$- \int_\Gamma \left(\widetilde{\boldsymbol{f}_\Gamma} \cdot \boldsymbol{v}_\Gamma + \widetilde{\boldsymbol{c}_\Gamma} \cdot \boldsymbol{w}_\Gamma\right) \mathrm{d}s - (\widetilde{\boldsymbol{n}_i} \cdot \boldsymbol{v}_{\Gamma i} + \widetilde{\boldsymbol{n}_e} \cdot \boldsymbol{v}_{\Gamma e}) \tag{7}$$
$$- (\widetilde{\boldsymbol{m}_i} \cdot \boldsymbol{w}_{\Gamma i} + \widetilde{\boldsymbol{m}_e} \cdot \boldsymbol{w}_{\Gamma e}) = 0 ,$$

where $\boldsymbol{v}_{\Gamma i}$, $\boldsymbol{w}_{\Gamma i}$ and $\boldsymbol{v}_{\Gamma e}$, $\boldsymbol{w}_{\Gamma e}$ are values of $\boldsymbol{v}_\Gamma$, $\boldsymbol{w}_\Gamma$ in the initial and end points of $\Gamma$, respectively.

Introducing (3)–(6) into (7) we can transform the identity as suggested in Chróścielewski et al. (2004), chapter 3. Then, if $\boldsymbol{v}$ and $\boldsymbol{w}$ are interpreted as the kinematically admissible virtual displacement vectors such that $\boldsymbol{v} = \boldsymbol{w} = \boldsymbol{0}$ along $\partial M_d = \partial M \setminus \partial M_f$, then (7) takes the form

$$\iint_{M\setminus\Gamma} (\boldsymbol{f} \cdot \boldsymbol{v} + \boldsymbol{c} \cdot \boldsymbol{w})\, \mathrm{d}a + \int_{\partial M_f} (\boldsymbol{n}^* \cdot \boldsymbol{v} + \boldsymbol{m}^* \cdot \boldsymbol{w})\, \mathrm{d}s$$
$$- \iint_{M\setminus\Gamma} \{\boldsymbol{N} \cdot (\mathrm{Grad}_s \boldsymbol{v} - \boldsymbol{W}\boldsymbol{F}) + \boldsymbol{M} \cdot \mathrm{Grad}_s \boldsymbol{w}\}\, \mathrm{d}a$$
$$- \int_\Gamma (\boldsymbol{f}_\Gamma \cdot \boldsymbol{v}_\Gamma + \boldsymbol{c}_\Gamma \cdot \boldsymbol{w}_\Gamma)\, \mathrm{d}s \tag{8}$$
$$+ \int_\Gamma \{\boldsymbol{n} \cdot (\boldsymbol{v}'_\Gamma - \boldsymbol{y}'_\Gamma \times \boldsymbol{w}_\Gamma) + \boldsymbol{m} \cdot \boldsymbol{w}'_\Gamma\}\, \mathrm{d}s$$
$$- \int_\Gamma \{[\boldsymbol{N}\nu] \cdot \boldsymbol{v}_\Gamma - [\boldsymbol{N}\nu \cdot \boldsymbol{v}] + [\boldsymbol{M}\nu] \cdot \boldsymbol{w}_\Gamma - [\boldsymbol{M}\nu \cdot \boldsymbol{w}]\}\, \mathrm{d}s$$
$$- (\boldsymbol{n}_e^* \cdot \boldsymbol{v}_{\Gamma e} - \boldsymbol{n}_i^* \cdot \boldsymbol{v}_{\Gamma i}) - (\boldsymbol{m}_e^* \cdot \boldsymbol{w}_{\Gamma e} - \boldsymbol{m}_i^* \cdot \boldsymbol{w}_{\Gamma i}) = 0 ,$$

where the scalar product of two tensors $(\boldsymbol{A}, \boldsymbol{B}) \in E \otimes T_x M$ is defined as $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}\left(\boldsymbol{A}^T\boldsymbol{B}\right)$, and for the skew tensor we have $\boldsymbol{W} = \boldsymbol{w} \times \boldsymbol{1}$, with $\boldsymbol{1}$ the unit tensor of $E \otimes E$.

Let shell displacements associated with a real deformation consist of a translation vector $\boldsymbol{u} = \boldsymbol{y} - \boldsymbol{x} \in E$ of $M$ and a rotation tensor $\boldsymbol{Q} \in SO(3)$ of the shell cross sections defined as $\boldsymbol{Q} = \boldsymbol{d}_i \otimes \boldsymbol{t}_i$, where $\boldsymbol{d}_i$, $i = 1, 2, 3$, and $\boldsymbol{t}_i = (\boldsymbol{t}_\alpha, \boldsymbol{t})$, $\alpha = 1, 2$, are triads of orthonormal directors in the deformed and undeformed placement, respectively. The virtual displacements can then be identified as $\boldsymbol{v} = \delta\boldsymbol{u}$ and $\boldsymbol{w} = \delta\boldsymbol{Q}\boldsymbol{Q}^T$, where $\delta$ is the symbol of virtual change (variation).

It can be shown that the exact 2D kinematic structure of $M \setminus \Gamma$ coincides with the one of the Cosserat surface and that of $\Gamma$ with the one of the Cosserat rod, see Cosserat & Cosserat (1909) and Chróścielewski et al. (2004), because some virtual expressions in (8) can be calculated as variations of the natural strain measures by

$$\mathrm{Grad}_s \boldsymbol{v} - \boldsymbol{W}\boldsymbol{F} = \delta^c \boldsymbol{E} , \quad \mathrm{Grad}_s \boldsymbol{w} = \delta^c \boldsymbol{K} ,$$
$$\boldsymbol{v}'_\Gamma - \boldsymbol{y}'_\Gamma \times \boldsymbol{w}_\Gamma = \delta^c \boldsymbol{\varepsilon}_\Gamma , \quad \boldsymbol{w}'_\Gamma = \delta^c \boldsymbol{\kappa}_\Gamma . \tag{9}$$

Here $\delta^c(\cdot) = \boldsymbol{Q}\{\delta(\boldsymbol{Q}^T(.))\}$ is the co-rotational variation of (.), and the strain measures of $M \setminus \Gamma$ and $\Gamma$ are defined by

$$\boldsymbol{E} = \boldsymbol{F} - \boldsymbol{Q}\boldsymbol{I} , \quad \boldsymbol{K} = \boldsymbol{C}\boldsymbol{F} - \boldsymbol{Q}\boldsymbol{B} ,$$
$$\boldsymbol{\varepsilon}_\Gamma = \boldsymbol{y}'_\Gamma - \boldsymbol{Q}\,\boldsymbol{t}_\Gamma , \quad \boldsymbol{\kappa}_\Gamma = \mathrm{ax}\left(\boldsymbol{Q}'_\Gamma\boldsymbol{Q}^T_\Gamma\right) , \tag{10}$$

where $\boldsymbol{I}$ is the inclusion operator of $M \setminus \Gamma$, while $\boldsymbol{C}$, $\boldsymbol{B}$ are the structure curvature tensors of the shell in the undeformed and deformed placement, respectively, see Eremeyev & Pietraszkiewicz (2006).

With (9) and (10) the integral identity (8) has the meaning of the principle of virtual work (PVW) for the branching shells. The first two lines of (8) are equivallent to the PVW of the regular shell suggested in somewhat different notation by Libai & Simmonds (1983). The remaining lines of (8) represent additional virtual work following from existence of the shell branching.

If we introduce the virtual strain energy densities in $M \setminus \Gamma$ and along $\Gamma$ defined as

$$\sigma = \boldsymbol{N} \cdot \delta^c \boldsymbol{E} + \boldsymbol{M} \cdot \delta^c \boldsymbol{K} \,, \quad \sigma_\Gamma = \boldsymbol{n} \cdot \delta^c \boldsymbol{\varepsilon}_\Gamma + \boldsymbol{m} \cdot \delta^c \boldsymbol{\kappa}_\Gamma \,, \quad (11)$$

then the PVW (8) for the branching shells can be given in the form

$$\int\!\!\int_{M\setminus\Gamma} (\boldsymbol{f} \cdot \delta \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{w}) \,\mathrm{d}a - \int\!\!\int_{M\setminus\Gamma} \sigma \,\mathrm{d}a$$
$$+ \int_{\partial M_f} (\boldsymbol{n}^* \cdot \delta \boldsymbol{u} + \boldsymbol{m}^* \cdot \boldsymbol{w}) \,\mathrm{d}s$$
$$- \int_\Gamma (\boldsymbol{f}_\Gamma \cdot \delta \boldsymbol{u}_\Gamma + \boldsymbol{c}_\Gamma \cdot \boldsymbol{w}_\Gamma) \,\mathrm{d}s + \int_\Gamma \sigma_\Gamma \,\mathrm{d}s \quad (12)$$
$$- \int_\Gamma \{[\boldsymbol{N}\nu] \cdot \delta \boldsymbol{u}_\Gamma - [\boldsymbol{N}\nu \cdot \delta \boldsymbol{u}] + [\boldsymbol{M}\nu] \cdot \boldsymbol{w}_\Gamma - [\boldsymbol{M}\nu \cdot \boldsymbol{w}]\} \,\mathrm{d}s$$
$$- (\boldsymbol{n}_e^* \cdot \delta \boldsymbol{u}_{\Gamma e} - \boldsymbol{n}_i^* \cdot \delta \boldsymbol{u}_{\Gamma i}) - (\boldsymbol{m}_e^* \cdot \boldsymbol{w}_{\Gamma e} - \boldsymbol{m}_i^* \cdot \boldsymbol{w}_{\Gamma i}) = 0 \,.$$

## 4 JUNCTIONS AT SHELL BRANCHING

To be more specific, let as discuss in more detail the branching shell consisting of three regular parts $M_k$, $k = 1, 2, 3$, joined together along the common junction modelled by the curve $\Gamma$, see Fig. 1. Let us assume that the base surface $M$ remains continuous during the deformation process, i.e. the translation field at $\Gamma$ satisfy the continuity conditions $\boldsymbol{u}_k = \boldsymbol{u}_\Gamma$, where $\boldsymbol{u}_k$ is the one-sided limit of $\boldsymbol{u}$ on $M_k$ when $\Gamma$ is approached. Then different types of junctions along $\Gamma$ can be described in terms of different constraints put on the one-sided limits $\boldsymbol{Q}_k$ of $\boldsymbol{Q}$ when $\Gamma$ is approached.

### 4.1 *Rigid junction*

The junction is rigid along $\Gamma$ if both $\boldsymbol{u}$ and $\boldsymbol{Q}$ are continuous at $\Gamma$, that is also $\boldsymbol{Q}_k = \boldsymbol{Q}_\Gamma$, $k = 1, 2, 3$, at $\Gamma$, see Fig. 2a). In this case $[\boldsymbol{N}\boldsymbol{\nu} \cdot \delta \boldsymbol{u}] = [\boldsymbol{N}\boldsymbol{\nu}] \cdot \delta \boldsymbol{u}_\Gamma$, $[\boldsymbol{M}\boldsymbol{\nu} \cdot \boldsymbol{w}] = [\boldsymbol{M}\boldsymbol{\nu}] \cdot \boldsymbol{w}_\Gamma$ and the integral $(12)_4$ identically vanishes. As a result, the kinematic structure of the branching shell with all junctions rigid along $\Gamma$ coincides with that of the regular Cosserat surface $M$ with the regular Cosserat curve $\Gamma$. Since $\boldsymbol{n}$, $\boldsymbol{m}$ along $\Gamma$ are generated by $\boldsymbol{n}_i^*$, $\boldsymbol{m}_i^*$ and $\boldsymbol{n}_e^*$, $\boldsymbol{m}_e^*$ given at the ends of $\Gamma$, the value of $\int_\Gamma \sigma_\Gamma \mathrm{d}s$ can easily be calculated for any geometry of $\Gamma$. As a result, kinematics of the branching shell is entirely defined by two fields $\boldsymbol{u}$, $\boldsymbol{Q}$ continuous on the whole $M$ containing $\Gamma$.

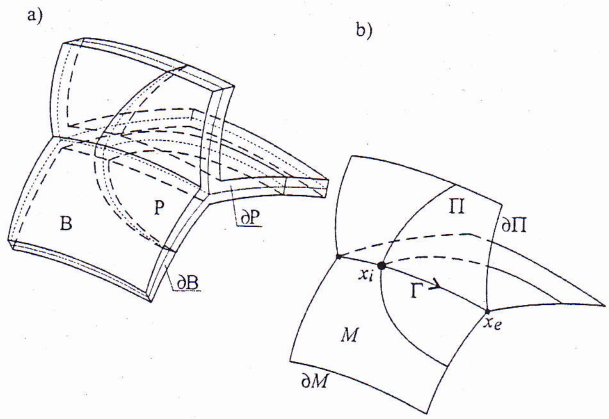


Figure 1. The branching shell element: a) the 3D shell, B) the corresponding 2D base surface.

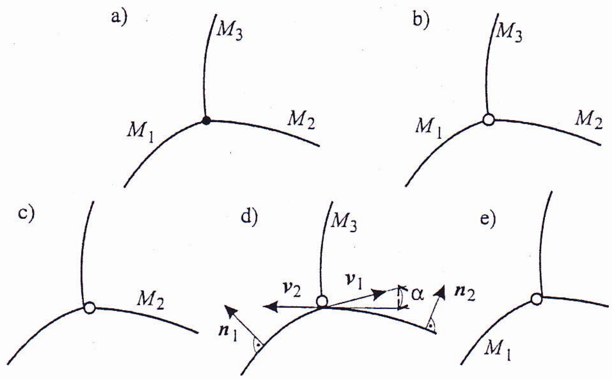


Figure 2. Junctions in the branching shell: a) rigid, b) entirely simply supported, c)-e) partly simply supported.

### 4.2 *Entirely simply supported junction*

The junction is entirely simply supported along $\Gamma$ if only $\boldsymbol{u}$ is continuous at $\Gamma$, but $\boldsymbol{Q}$ is not constrained at all when approaching $\Gamma$ on $M_k$, $k = 1, 2, 3$. It means that approaching $\Gamma$ we have to satisfy three independent static continuity conditions $\boldsymbol{M}_k \boldsymbol{\nu}_k = \boldsymbol{0}$, see Fig. 2b). Then the relation $[\boldsymbol{N}\boldsymbol{\nu} \cdot \delta \boldsymbol{u}] = [\boldsymbol{N}\boldsymbol{\nu}] \cdot \delta \boldsymbol{u}_\Gamma$ still holds, the third term of $(12)_4$ identically vanishes, the forth term of $(12)_4$ also vanishes because $[\boldsymbol{M}\boldsymbol{\nu} \cdot \boldsymbol{w}] = \boldsymbol{0}$, and the curvilinear integral $(12)_4$ over $\Gamma$ vanishes as well. In this ease the rotation field $\boldsymbol{Q}_\Gamma$ becomes undefinable relative to any $\boldsymbol{Q}_k$ on $M \setminus \Gamma$.

### 4.3 *Partly simply supported junction*

The junction can be called partly simply supported along $\Gamma$ if $\boldsymbol{u}$ is continuous at $\Gamma$, one $\boldsymbol{Q}_k$ is not constrained while the remaining two $\boldsymbol{Q}_k$ are assumed to coincide with $\boldsymbol{Q}_\Gamma$ when $\Gamma$ is approached. Let us, for definiteness, discuss the case sketched in Fig. 2d). Then the continuity conditions along $\Gamma$ become

$$\boldsymbol{M}_3 \boldsymbol{\nu}_3 = \boldsymbol{0} \,, \quad \boldsymbol{Q}_1 = \boldsymbol{Q}_2 = \boldsymbol{Q}_\Gamma \,. \quad (13)$$

To be more specific, one has to introduce along $\Gamma$ the orthonormal triad $\boldsymbol{\nu}_\Gamma$, $\boldsymbol{\tau}_\Gamma$, $\boldsymbol{n}_\Gamma$ with $\boldsymbol{\tau}_\Gamma$ tangent to $\Gamma$ in the positve direction, see Fig. 1b). Then choosing orientations of $M_1$ and $M_2$ described by the unit normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ and taking $\boldsymbol{n}_\Gamma = \boldsymbol{n}_2|_\Gamma$, as in Fig. 2d), we may relate the respective $\nu_1$ and $\nu_2$ to the common $\nu_\Gamma$. Since in this case $\tau_1 = -\tau_\Gamma$, $\tau_2 = +\tau_\Gamma$, we may choose for example $\nu_2 = \nu_\Gamma$ and then $\nu_1 = -\nu_\Gamma \cos\alpha$. In such a case, according to (13) we obtain

$$[M\nu \cdot w] = \{(M_2 - M_1 \cos\alpha)\, \nu_\Gamma\} \cdot w_\Gamma = [M\nu] \cdot w_\Gamma \,, \qquad (14)$$

so that the curvilinear integral $(12)_4$ vanishes again leading to the same form of the PVW as for the rigid and entirely simply supported junctions. However, now $Q_\Gamma$ is defined in $(13)_2$ while $Q_3$ may be found only in the process of solution.

### 4.4 *Partly deformable junction*

Besides of simple types of junctions discussed above, there may be a number of other types of junctions defined by assuming various combinations of constraints put on components of $u_k$ and $Q_k$ relative to appropriately defined $u_\Gamma$ and $Q_\Gamma$. In particular, these may be elastically deformed junctions defined by $N_k \nu_k = C_k u_\Gamma$ and/or $M_k \nu_k = D_k \phi_\Gamma$, where $C_k$, $D_k$ are given 2nd-order tensors, while $\phi_\Gamma = \phi_\Gamma e_\Gamma$ is the finite rotation vector corresponding to $Q_\Gamma$ such that $Q_\Gamma = \exp(\phi_\Gamma \times 1)$. But the tensors $C_k$, $D_k$ themselves may be assumed to depend on $u_k$, $Q_k$ to model non-linear behaviour of the junction and possibly also on $\delta u_k$, $\delta Q_k Q_k^T$ to model dissipative effects. Each combination of such constraints put on particular components of $u_k$, $Q_k$ should be annalysed separately, and in each case the curvilinear integral $(12)_4$ may be reduced to a different expression leading to a different form of the PVW for this particular type of partly deformable junction.

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