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On displacemental version of the non–linear theory of thin shells

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ABSTRACT: We propose the modified version of the non-linear theory of thin shells expressed in terms of displacements of the shell reference surface as the only independent field variables. In our approach the final displacemental boundary value problem (BVP) and associated homogeneous shell buckling problem (SBP) are generated exactly only in the computer memory by two packages set up in MATHEMATICA. This approach allows us to account also for those a few supposedly small terms, which may be critical for finding the correct buckling load of shells sensitive to imperfections.

1 INTRODUCTION

The entirely Lagrangian non-linear theory of thin elastic shells, expressed in terms of displacements $\mathbf{u}$ of the shell reference surface as the only independent field variables, was first proposed by Pietraszkiewicz & Szwabowicz (1981) and developed by Pietraszkiewicz (1984), where references to earlier attempts in the field were given. The formulation followed from the principle of virtual work (PVW) postulated for the shell base surface.

Our recent experience gained while writing three reports by opoka & Pietraszkiewicz (2004, 2009a,b) allows us to propose in this paper the following three modifications of the non-linear displacemental shell equations:

- The vector equilibrium equations of Pietraszkiewicz (1984) are represented through components in the contravariant base vectors of the deformed reference surface, and the tangential scalar equilibrium equations are exactly simplified using the compatibility conditions.
- Along the boundary contour of the reference surface the new scalar function α rational with regard to displacement derivatives is defined, and the new sets of four work-conjugate static and geometric boundary conditions are derived.
- For any definite geometry of the reference surface parameterized by orthogonal coordinates and for any of its boundaries, the displacemental BVP and SBP are generated automatically and exactly by the use of two packages *ShellGeom.m* and *ShellBVP.m* set up in MATHEMATICA.

In our approach we do not simplify the shell relations in the process of expressing the surface stress and strain measures in terms of displacements. As a result, the displacemental BVP and the associated SBP become extremely complex and not tractable by hand transformations; they appear only as relations generated directly in the computer memory. This allows one to account for those a few supposedly small terms in the SBP, which may be critically important for finding the correct buckling load of shell structures sensitive to imperfections.

2 NOTATION AND KINEMATIC RELATIONS

In the undeformed configuration the shell reference surface $\mathcal{M}$ is given by the position vector $r = \mathbf{r}(\theta^\alpha)$ relative to a point $O \in \mathcal{E}$. The geometry of $\mathcal{M}$ is described by the covariant base vectors $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$, the covariant components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ of the surface metric tensor $\mathbf{a}$ with $a = \det(a_{\alpha\beta}) > 0$, the contravariant components $\varepsilon^{\alpha\beta}$ of the surface permutation tensor $\boldsymbol{\varepsilon}$, the unit normal vector $\mathbf{n} = \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ orienting $\mathcal{M}$, and the covariant components $b_{\alpha\beta} = -\mathbf{a}_{,\alpha} \cdot \mathbf{n}_{,\beta} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta}$ of the surface curvature tensor $\mathbf{b}$. The boundary contour $\partial\mathcal{M}$ of $\mathcal{M}$ consists of a finite number of piecewise smooth curves given by $\mathbf{r}(s) = \mathbf{r}[\theta(s)]$, where s is the arc-length along $\partial\mathcal{M}$. With each regular point $M \in \partial\mathcal{M}$ we associate the unit tangent vector $\boldsymbol{\tau} \equiv \mathbf{r}_{,s} = d\mathbf{r}/ds = \tau^\alpha \mathbf{a}_\alpha$, and the outward unit normal vector $\boldsymbol{\nu} = \boldsymbol{\tau} \times \mathbf{n} = \nu^\alpha \mathbf{a}_\alpha$. For other geometric definitions and relations we refer to Pietraszkiewicz (1977).

The deformed configuration $\bar{\mathcal{M}}$ of the surface $\mathcal{M}$ can be described by the position vector $\bar{\mathbf{r}}(\theta^\alpha) = \mathbf{r}(\theta^\alpha) + \mathbf{u}(\theta^\alpha)$ relative to the same point $O \in \mathcal{E}$, where θ^α are the surface curvilinear convected coordinates, and $\mathbf{u} = u_\alpha \mathbf{a}^\alpha + u_3 \mathbf{n}$ is the displacement field. In convected coordinates geometric quantities and relations on the deformed surface $\bar{\mathcal{M}}$ are defined analogously as their counterparts in the undeformed configuration; they will be marked here by an additional dash, for example $\bar{\mathbf{a}}_\alpha$, $\bar{a}^{\alpha\beta}$, $\bar{b}_{\alpha\beta}$, $\bar{\mathbf{n}}$, $\bar{\boldsymbol{\nu}}$, etc. All dashed fields on $\bar{\mathcal{M}}$ can be expressed through analogous undashed fields defined on $\mathcal{M}$ and the displacement field $\mathbf{u}$, see for example Pietrazkiewicz (1984, 1989). In particular, we have

$$
\begin{aligned}
&\bar{\mathbf{a}}_\alpha = l_{\lambda\alpha}\mathbf{a}^\lambda + \varphi_\alpha \mathbf{n}\,, \qquad \bar{\mathbf{n}} = \sqrt{\tfrac{a}{\bar a}}\left(m_\lambda \mathbf{a}^\lambda + m\mathbf{n}\right),\\
&\varphi_\alpha = u_{3,\alpha} + b^\lambda_\alpha u_\lambda\,, \qquad l_{\lambda\alpha} = a_{\lambda\alpha} + u_\lambda|_\alpha - b_{\lambda\alpha}u_3\,,\\
&m_\lambda = \varphi_\alpha l^\alpha_{.\lambda} - \varphi_\lambda l^\alpha_{.\alpha}\,, \qquad m = \tfrac{1}{2}\left(l^\alpha_{.\alpha}l^\beta_{.\beta} - l^\beta_{.\alpha}l^\alpha_{.\beta}\right).
\end{aligned} \tag{1}
$$

Components of the symmetric surface strain and bending measures of the Green type are defined by the relations

$$
\begin{aligned}
&\gamma_{\alpha\beta} = \frac{1}{2}\left(\bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{a}}_\beta - a_{\alpha\beta}\right) = \frac{1}{2}\left(l^\lambda_{.\alpha}l_{\lambda\beta} + \varphi_\alpha\varphi_\beta - a_{\alpha\beta}\right),\\
&\kappa_{\alpha\beta} = b_{\alpha\beta} - \bar{\mathbf{n}}\cdot\bar{\mathbf{a}}_\alpha|_\beta = b_{\alpha\beta} - \sqrt{\tfrac{a}{\bar a}}\chi_{\alpha\beta}\,,\\
&\chi_{\alpha\beta} = m\left(\varphi_\alpha|_\beta + b_{\lambda\beta}l^\lambda_{.\alpha}\right) + m_\lambda\left(l^\lambda_{.\alpha}|_\beta - b^\lambda_\beta\varphi_\alpha\right).
\end{aligned} \tag{2}
$$

3 MODIFIED EQUILIBRIUM EQUATIONS

Under some kinematic assumptions summarised by Pietraszkiewicz (1989) for the geometrically non-linear theory of elastic shells and proposed by Schieck et al. (1992) for the large-strain theory of rubber-like shells, or alternately under the constitutive assumptions proposed by Libai & Simmonds (1998), the mechanical behaviour of a thin shell is entirely described by stretching and bending of its reference surface.

Let $\bar{\mathscr{M}}$ be the reference surface of the deformed shell in an equilibrium state under the surface force $\mathbf{p}(\theta^\alpha) = p_\alpha\bar{\mathbf{a}}^\alpha + p\bar{\mathbf{n}}$ and couple $\mathbf{c}(\theta^\alpha) = \bar{\mathbf{n}}\times c^\alpha\bar{\mathbf{a}}_\alpha$ vectors, both measured per unit area of the reference surface $\mathscr{M}$, and under the boundary force $\mathbf{N}^*(s) = N^*_\nu\boldsymbol{\nu} + N^*_\tau\boldsymbol{\tau} + N^*\mathbf{n}$ and couple $\mathbf{M}^*(s) = \bar{\mathbf{n}}\times(M^*_\nu\bar{\boldsymbol{\nu}} + M^*_\tau\bar{\boldsymbol{\tau}})$ vectors, both measured per unit length of the undeformed boundary contour $\partial\mathscr{M}$. Then, for all kinematically admissible virtual displacements $\delta\mathbf{u}$ the equilibrium conditions for $\bar{\mathscr{M}}$ are given by the principle of virtual work (PVW)

$$
\begin{aligned}
&\iint_{\mathscr{M}} \left(N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta}\right) dA\\
&= \iint_{\mathscr{M}} \left(\mathbf{p}\cdot\delta\mathbf{u} + \mathbf{c}\cdot\boldsymbol{\omega}\right) dA + \int_{\partial\mathscr{M}_f} \left(\mathbf{N}^*\cdot\delta\mathbf{u} + \mathbf{M}^*\cdot\boldsymbol{\omega}_\tau\right) ds,
\end{aligned} \tag{3}
$$

where $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are components of the symmetric surface stress resultant and stress couple tensors of the Kirchhoff type, $\delta\gamma_{\alpha\beta}$ and $\delta\kappa_{\alpha\beta}$ are virtual changes of the strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$, while $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_\tau$ are the virtual rotation vectors at $\mathscr{M}$ and along $\partial\mathscr{M}$, respectively.

Using the Stokes theorem in $\mathscr{M}$, the PVW can be transformed into an alternative form from which follows the known vector equilibrium equation

$$
\mathbf{T}^\beta|_\beta + \mathbf{p} + (c^\beta\bar{\mathbf{n}})|_\beta = \mathbf{0} \ \text{ in } \mathscr{M}\,, \tag{4}
$$

where

$$
\mathbf{T}^\beta = \left(N^{\alpha\beta} - \bar b^\alpha_\lambda M^{\lambda\beta}\right)\bar{\mathbf{a}}_\alpha + \left(M^{\alpha\beta}|_\alpha + \bar a^{\beta\kappa}\gamma_{\kappa\lambda\mu}M^{\lambda\mu}\right)\bar{\mathbf{n}}. \tag{5}
$$

From (4) we obtain three scalar equilibrium equations in the deformed surface contravariant base, which are expressed through the mixed components of the surface stress and strain measures

$$
\begin{aligned}
&\underline{N^\beta_\alpha|_\beta} + 2\gamma^\lambda_\alpha N^\beta_\lambda|_\beta + \left(2\gamma^\lambda_\alpha|_\beta - \gamma^\lambda_\beta|_\alpha\right)N^\beta_\lambda - 2\left(b^\lambda_\alpha - \kappa^\lambda_\alpha\right)M^\beta_\lambda|_\beta\\
&+ \left[(b^\lambda_\beta - \kappa^\lambda_\beta)|_\alpha - 2\left(b^\lambda_\alpha - \kappa^\lambda_\alpha\right)|_\beta\right]M^\beta_\lambda - \left(b^\lambda_\alpha - \kappa^\lambda_\alpha\right)c_\lambda + \underline{p_\alpha} = 0\,,\\
&\left\{\frac{a}{\bar a}\left[\left(1 + 2\gamma^\mu_\mu\right)\left(2\gamma^\alpha_\beta|^\lambda - \gamma^\lambda_\beta|^\alpha\right) - 2\gamma^\alpha_\mu\left(2\gamma^\mu_\beta|^\lambda - \gamma^\lambda_\beta|^\mu\right)\right]M^\beta_\lambda\right\}\Big|_\alpha\\
&- \frac{a}{\bar a}\left[\left(1 + 2\gamma^\mu_\mu\right)\left(b^\alpha_\beta - \kappa^\alpha_\beta\right) - 2\gamma^\alpha_\mu\left(b^\mu_\beta - \kappa^\mu_\beta\right)\right]\left(b^\lambda_\alpha - \kappa^\lambda_\alpha\right)M^\beta_\lambda\\
&+ \underline{M^\alpha_\beta|^\beta_\alpha + \left(b^\beta_\alpha - \kappa^\beta_\alpha\right)N^\alpha_\beta + c^\alpha|_\alpha + p} = 0\,.
\end{aligned} \tag{6}
$$

4 MODIFIED WORK-CONJUGATE BOUNDARY CONDITIONS

Along $\partial\mathscr{M}$ the virtual work performed by $\mathbf{M}^*$ on $\boldsymbol{\omega}_\tau$ can equivalently be expressed as

$$
\mathbf{M}^*\cdot\boldsymbol{\omega}_\tau = \mathbf{H}^*\cdot\delta\bar{\mathbf{n}}\,, \quad \mathbf{H}^* = M^*_\nu\bar{\boldsymbol{\nu}} + M^*_\tau\bar{\boldsymbol{\tau}}\,. \tag{7}
$$

From our numerical experience gained in the recent report by opoka & Pietraszkiewicz (2009b), in this paper we introduce along $\partial\mathscr{M}$ the new scalar function of displacement (or position) derivatives defined by

$$
\alpha = \frac{m_\nu}{m} = \frac{\varphi_\tau l_{\tau\nu} - \varphi_\nu l_{\tau\tau}}{l_{\nu\nu}l_{\tau\tau} - l_{\nu\tau}l_{\tau\nu}} = \alpha(\mathbf{u}_{,\nu}\,, \mathbf{u}_{,s})\,. \tag{8}
$$

The formula for $\bar{\mathbf{n}}$ in terms of α and $\mathbf{u}_{,s}$ becomes

$$
\bar{\mathbf{n}} = \sqrt{\frac{a}{\bar a}}\,m\left[\alpha\boldsymbol{\nu} - \frac{1}{l_{\tau\tau}}\left(l_{\nu\tau}\alpha + \varphi_\tau\right)\boldsymbol{\tau} + \mathbf{n}\right]. \tag{9}
$$

Variating the expression (8) to have

$$
\delta\alpha = \frac{1}{m^2}\left(l_{\tau\nu}\bar{\mathbf{m}}\cdot\delta\mathbf{u}_{,s} - l_{\tau\tau}\bar{\mathbf{m}}\cdot\delta\mathbf{u}_{,\nu}\right), \tag{10}
$$

and calculating $\bar{\mathbf{m}}\cdot\delta\mathbf{u}_{,\nu}$ from (10), we obtain the new relation for $\delta\bar{\mathbf{n}}$ at $\partial\mathscr{M}$

$$
\delta\bar{\mathbf{n}} = \bar{\mathbf{a}}^\beta\left[\sqrt{\frac{a}{\bar a}}\frac{m^2}{l_{\tau\tau}}\nu_\beta\delta\alpha - \left(\frac{l_{\tau\nu}}{l_{\tau\tau}}\nu_\beta + \tau_\beta\right)\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,s}\right]. \tag{11}
$$

The expressions (11) and (7) can now be used in the second term of the last line integral in the PVW. After integration by parts the line integral can be transformed to an alternative form from which follow the natural static boundary and corner conditions

$$
\begin{aligned}
&\mathcal{C}^\alpha\bar{\mathbf{a}}_\alpha + \mathcal{D}\bar{\mathbf{n}} = \mathbf{N}^* - \bar a^{\alpha\mu}\bar b_{\mu\lambda}\tau^\lambda\mathcal{F}^*\bar{\mathbf{a}}_\alpha + \mathcal{F}^*{}_{,s}\bar{\mathbf{n}}\,,\\
&M_{\nu\nu} = a_\tau\sqrt{\tfrac{a}{\bar a}}M^*_\nu \ \text{ on } \partial\mathscr{M}_f\,,
\end{aligned} \tag{12}
$$

$$\mathcal{F}\bar{n} = \mathcal{F}^*\bar{n} \text{ at each corner } C_i \in \partial\mathcal{M}_f . \quad (13)$$

In (12) we have introduced

$$\mathcal{C}^\alpha = \underline{N^{\alpha\beta}\nu_\beta} - \bar{a}^{\alpha\mu}\bar{b}_{\mu\lambda}\left(M^{\lambda\beta}\nu_\beta + \tau^\lambda \mathcal{F}\right) ,$$

$$\mathcal{D} = \underline{M^{\alpha\beta}|_\beta\, \nu_\alpha + \mathcal{F}_{,s}} + \bar{a}^{\alpha\mu}\gamma_{\mu\lambda\beta}M^{\lambda\beta}\nu_\alpha + \underline{c^\alpha\nu_\alpha} ,$$

$$\mathcal{F} = \frac{l_{\tau\nu}}{l_{\tau\tau}}M_{\nu\nu} + M_{\nu\tau} , \quad (14)$$

$$\mathcal{F}^* = \frac{M^*_\tau}{a_\tau} - \sqrt{\frac{a}{\bar{a}}}\frac{l_{\nu\tau}m - \varphi_\tau m_\nu}{a_\tau l_{\tau\tau}}M^*_\nu .$$

All fields present in the boundary conditions (12) are functions of the arc-length coordinate s of $\partial\mathcal{M}$.

The geometric boundary conditions which are work-conjugate to the static ones (12) are

$$\mathbf{u} = \mathbf{u}^*, \quad \alpha(\mathbf{u}_{,\nu}, \mathbf{u}_{,s}) = \alpha^* \text{ on } \partial\mathcal{M}_d . \quad (15)$$

5 MODIFIED BVP AND SBP IN TERMS OF DISPLACEMENTS

To formulate the BVP in terms of displacements, the stress measures N^β_α and M^β_α should be eliminated from (6) and (12) by the constitutive equations, and then the strain measures γ^β_α and κ^β_α should be expressed through displacements using the strain-displacement relations (2). We briefly discuss below such BVPs for two simple cases of the constitutive equations.

Within the first-approximation theory of thin shells made of a homogeneous, isotropic, elastic material undergoing small strains, the strain energy density of the shell is given by

$$\Sigma = \frac{h}{2}H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}\right) + O(Eh\eta^2\epsilon) , \quad (16)$$

and the corresponding constitutive equations are

$$N^{\alpha\beta} = \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} , \quad M^{\alpha\beta} = \frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}} , \quad (17)$$

where $H^{\alpha\beta\lambda\mu}$ are components of the modified elasticity tensor, η denotes the maximal strain in the shell space and ϵ describes formally the energetic error of this shell theory (Koiter 1960). If we substitute (17) into the BVP and reject all terms of the order of error introduced by the constitutive equations (17), then only the underlined terms in (6) and $(14)_{1,2}$ remain as the primary important terms, and $\sqrt{\frac{a}{\bar{a}}} \simeq 1$, $a_\tau \simeq 1$.

Another example of the theory of shells based on the PVW (3) is the large-strain bending theory of elastic rubber-like shells. Various versions of such a theory were proposed in the literature. In particular, when the greater eigenvalue γ of $\gamma_{\alpha\beta}$ was additionally assumed to be at most moderate, so that the approximation $1 + \gamma^2 \approx 1$ holds, the strain energy density follows from that proposed by Schieck et al. (1992) in possibly the simplest form

$$\Sigma = hW_{(0)}(\gamma_{\kappa\rho}) + \frac{h^3}{24}W^{\alpha\beta\lambda\mu}_{(2)}(\gamma_{\kappa\rho})[\kappa_{\alpha\beta}\kappa_{\lambda\mu}(1 - \gamma^g_\sigma)] , \quad (18)$$

where $W_{(0)}$ and $W^{\alpha\beta\lambda\mu}_{(2)}$ are the 3D strain energy density and its second derivative relative to $\gamma_{\kappa\rho}$, both taken at $\mathcal{M}$. Then the corresponding constitutive equations follow again from (17).

Using the estimate for moderate surface strains and other assumptions made while deriving (18), it is also possible to considerably simplify the equilibrium equations (6) and boundary conditions (12) with (14) by omitting many supposedly small terms of the same order, if necessary.

However, in the present paper we take a radically different approach. Accepting our inability to reasonably select a few critically important small terms in the shell equilibrium conditions among many other small terms which can be ignored, we do not simplify the shell relations at all in the process of elimination of the surface stress and strain measures. Due to enormous complexity of the resulting displacement shell relations, such a BVP and the associated SBP have been derived with the help of two packages *ShellGeom.m* and *ShellBVP.m* written within the symbolic programming language of MATHEMATICA.

For the specified position vector $\mathbf{r} = \mathbf{r}(\theta^\alpha)$ the package *ShellGeom.m* generates all geometric characteristics of the undeformed shell reference surface $\mathcal{M}$ needed in transforming tensorial BVP to that expressed in partial derivatives. If the surface has a boundary contour $\partial\mathcal{M}$ the package additionally generates all necessary boundary characteristics.

For the specified system of orthogonal coordinates $\{\theta^1, \theta^2\}$ and the specified constitutive equations (17) the package *ShellBVP.m* generates the displacemental BVP and the asociated SBP. The output of this package is extremely large, because no approximation is used during generation of the BVP. Thus, the resulting BVP and SBP are available only in the computer memory.

The components of external loads $\mathbf{p}$, $\mathbf{c}$ and $\mathbf{N}^*$, $\mathbf{M}^*$ may be specified entirely independently, in general, by ten dimensionless parameters ρ_p forming the vector $\boldsymbol{\rho} \in \mathbf{R} \subset \mathbb{R}^{10}$. Then the non-linear BVP generated by the package *ShellBVP.m* can be presented symbolically as

$$f(\mathbf{u}; \boldsymbol{\rho}) = 0 , \quad (19)$$

where the non-linear continuosly differentiable operator f is defined on the product space $\mathscr{C}(\mathcal{M}, \mathbb{R}^3)$ of all components of $\mathbf{u}$ and its gradients up to the 4th order. In engineering applications all the external loads are usually specified by a single common parameter $\rho \in R \subset \mathbb{R}$, $\rho \geq 0$.

The solutions $\mathbf{u}_0(\rho)$ of (19), which can be reached starting form $\rho = 0$ in the undeformed state, form the primary equilibrium path. This path becomes unstable if an infinitesimally close adjacent equilibrium state $\mathbf{u}_1(\rho)$ exists for the same value of ρ.

In the neighbourhood of critical values of ρ we can replace $\mathbf{u}_1(\rho)$ by $\mathbf{u}_0(\rho)+\mathbf{u}$, where now $\mathbf{u}$ denotes the small increment of the displacements satisfying homogeneous boundary conditions. As a result, we obtain the homogeneous linear SBP in terms of the incremental displacements $\mathbf{u}$ which can be written again symbolically as

$$g(\mathbf{u};\rho) = 0\,. \tag{20}$$

Non-trivial solutions of (20) can exist only at a discrete set of values of ρ, which are eigenvalues of the linear SBP (20). The lowest positive eigenvalue $\rho_1 \equiv \rho_{crit}$ indicates the first bifurcation point at with the primary equilibrium path $\mathbf{u}_0(\rho)$ is intersected by a secondary equilibrium path $\mathbf{u}_1(\rho)$.

In the paper by Opoka & Pietraszkiewicz (2009b, Appendix) we present in more detail the derivation of the BVP (19) and the corresponding SBP (20) for the membrane prebuckling state in the special case of axially compressed circular cylinder. We also perform there the extensive numerical analysis of bifurcation buckling for a wide range of length-to-diameter ratios of the cylinder under fourteen sets of work–conjugate boundary conditions. The results are summarised in another our paper submitted to this conference.

6 CONCLUSIONS

We have formulated a new version of the Lagrangean non–linear theory of thin shells expressed in terms of displacements of the shell reference surface as the only independent field variables. The formulation has been based on the principle of virtual work postulated for the reference surface. Both the equilibrium equations and the set of four work–conjugate static and geometric boundary conditions are derived exactly from the PVW without using any kind of approximations. Elimination of the surface stress and strain measures in terms of displacements is performed exactly as well without using the approximate nature of the constitutive equations to simplify the BVP. The latter steps have been performed automatically with the help of two packages set up in MATHEMATICA. The final BVP and the corresponding SBP are obviously extremely complex. They are manageable only as the relations given in the computer memory, not as those explicitly written on the paper. By taking into account all supposedly small terms in the buckling shell equations we are sure that among them are also those a few supposedly small terms which may appear to be critically important ones in finding the correct buckling load of thin shells sensitive to imperfections.

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