On modelling and non-linear elasto-plastic analysis of thin shells with deformable junctions

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The undeformed base surface of the irregular thin shell is modelled by the union of a finite number of regular smooth surface elements joined together along spatial curvilinear surface edges. The equilibrium conditions are formulated by postulating an appropriate form of the principle of virtual work, where also deformability of shell junctions is taken into account. The PVW is then discretised by C^1 finite elements and the incremental-iterative procedure is applied to solve the highly non-linear BVP. As an example, the axisymmetric deformation state is calculated in the casing of devise measuring the external pressure and having two pairs of circular welded junctions. The problem is solved within the elasto-plastic range of material behaviour with linear combination of isotropic and kinematic hardening, and deformability of the junctions is taken into account.

1 Introduction

Thin-walled shell structures often consist of several regular shell elements interconnected along curvilinear junctions. Pressure vessels, silos, liquid and gas storage tanks, offshore platforms, tubular towers, branching and intersecting pipelines, rockets, multi-segment underwater pressure hulls are just a few examples of such irregular shell structures. In most of such multi-shells the adjacent elements are usually stiffly connected to each other. But in some cases the shell junctions may have their own mechanical properties allowing the adjacent shell elements to deform (usually rotate) one with respect to the other. Such junctions are called deformable here.

In the literature, there is a number of papers on axisymmetric behaviour of structures consisting of two or more thin shells of revolution (cylinders, cones, spheres, toroids) stiffly connected along the common circular junctions. These analyses performed within the classical linear theory of shells were partly summarised for example by Chernykh [6], Calladine [3], Baker et al. [1], and Novozhilov et al. [19]. The reviews of recent research on strength and stability of axisymmetric steel shell intersections was presented by Teng [28,29]. More general cases of intersecting shells within the linear elastic material behaviour were discussed for example by Voloshin and Samsonov [31], Bernadou [2], Ciarlet [9], Skopinsky [25-27] and Xue et al. [32,33]. But deformability of the junction itself according to the simplest model of linearly elastic behaviour was accounted for only by Bernadou [2], Nardinocci [17] and Titeux and Sanchez-Palencia [30] when discussing junctions of the linearly elastic plates.

The non-linear theory of thin irregular shells was developed in [15,16]. The irregular shell structure is modelled by a material irregular base surface-like continuum which is capable of resisting only to stretching and bending. In the undeformed configuration the surface-like continuum is a union of regular smooth surface elements joined together along spatial curvilinear surface edges. Each spatial curve can represent a singular surface curve, but also a one-dimensional continuum modelling the deformable junction. The behaviour of the whole irregular shell structure

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is formulated in the Lagrangian description by postulating an appropriate form of the principle of virtual work (PWV) in terms of displacements of the base surface-like continuum as the only independent field variables. Besides the equilibrium equations and boundary conditions known from the theory of thin regular shells, several forms of static and kinematic jump conditions at the junction are proposed in [16]. The jump conditions constitute an integral part of the boundary value problem (BVP). Simplified forms of the jump conditions appropriate for simpler models of thin irregular shells are given in [8].

Deformability of the junction is modelled in [16] implicitly by allowing appropriate forms of the one-dimensional virtual strain energy density. Unfortunately, we are not aware of any representative non-linear numerical example of thin irregular shells with deformable junctions, where the jump conditions and the effects of junction deformability itself would be tested. These remark are in sharp contrast to analyses of one-dimensional steel framed structures, where various semi-rigid beam-to-column connections were discussed in a number of papers, summarised in several recent books such as [5,10,4,12] and have even been introduced into Eurocode 3, [10].

In this paper we first recall in Section 2 the theoretical formulation of [16] and extend it slightly at the junction by explicit account of the junction deformability in the principle of virtual work. Then in Section 3 following the books [13,24,20] we briefly remind the incremental-iterative solution procedure of the non-linear boundary value problem, which is formulated for the whole irregular shell structure with deformable junctions. For any known approximation to the equilibrium state, an explicit form of incremental relations for the correcting increment of the displacements can be derived just by extending those presented in [22]. However, in this paper we do not derive explicitly all ingredients of such an incremental-iterative procedure, but perform all appropriate transformations implicitly within the numerical code. Such an approach seems to be more efficient for numerical analysis of this highly non-linear problem of irregular shells.

In order to evaluate the effectiveness of such modelling of elasto-plastic shell behaviour with deformable junctions in a simple one-dimensional case, an appropriate algorithm based on the computer code MINIMOD described in [7] is developed. The algorithm uses the C^1 axisymmetric displacemental finite elements applied within the elasto-plastic shell behaviour, which is extended here to account the deformable junctions. Numerical tests are performed for an axisymmetric non-linear behaviour of the casing of measuring device subjected to external pressure. The casing consists of the cylindrical part joined with two toroidal caps connected at boundaries with other parts through welding modelled here as either stiff or simply connected junctions. The influence of the type of junction stiffness on the stress distribution in the casing is discussed within the elasto-plastic range of material behaviour of the shell.

2 Notation and basic relations for irregular shells

According to [15,16], the consistent field equations and jump conditions of thin irregular shell structures can be derived using two postulates. In the kinematic one, the deformation of the irregular shell is assumed to be determined by stretching and bending of the irregular surface-like material continuum being a union of regular smooth surface elements joined together along spatial curvilinear surface edges, which in the reference configuration are denoted as M and Γ, respectively. Then the equilibrium conditions are required to be derivable from the principle of virtual work (PVW) involving only dynamic fields associated with the assumed kinematics of M. Such PVW is postulated in the form

$$G \equiv G_i - G_e - G_\Gamma = 0, \tag{1}$$

where G_i means the internal virtual work, G_e is the external virtual work, and G_Γ accounts for an additional virtual work of forces and couples acting only along all singular curves modelling the shell junctions.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be the position vectors of the shell base surface-like continuum in undeformed M and deformed $\bar{M} = \chi(M)$ configurations, respectively, such that $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$, with χ denoting deformation and $\boldsymbol{u}$ the translation vector field on M. Let $\boldsymbol{G} = \boldsymbol{P}\mathrm{Grad}_s\chi$ be the surface deformation gradient, with $\boldsymbol{P}$ the perpendicular projection on the tangent plane $T_{\boldsymbol{y}}\bar{M}$. Then two symmetric tensors

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{G}^T\bar{\boldsymbol{A}}\boldsymbol{G} - \boldsymbol{A}\right), \quad \boldsymbol{K} = -\left(\boldsymbol{G}^T\bar{\boldsymbol{B}}\boldsymbol{G} - \boldsymbol{B}\right) \tag{2}$$

are the Green-type strain and bending measures of any regular surface element $M^{(k)} \subset M$. In (2), $\boldsymbol{A},\boldsymbol{B}$ and $\bar{\boldsymbol{A}},\bar{\boldsymbol{B}}$ are the surface metric and curvature tensors in the undeformed and deformed configuration, respectively. In the Lagrangian description G_i in (1) can now be written as

$$G_i = \sum_k \iint_{M^{(k)}} \left(\boldsymbol{N}:\delta\boldsymbol{E} + \boldsymbol{M}:\delta\boldsymbol{K}\right)da, \tag{3}$$

where $\boldsymbol{N},\boldsymbol{M}$ are the surface symmetric stress resultant and stress couple tensors of the 2nd Piola - Kirchhoff type, δ is the symbol of variation, the double-dot $:$ means the scalar product in the tensor space $T_{\boldsymbol{x}}M \otimes T_{\boldsymbol{x}}M$ such that $\boldsymbol{S}:\boldsymbol{T} = \mathrm{tr}(\boldsymbol{S}^T\boldsymbol{T})$ for any surface tensors $\boldsymbol{S},\boldsymbol{T}$, and da means the elementary surface element of M.

Let $\boldsymbol{p}$ and $\boldsymbol{h}$ be the external surface force and moment resultant vectors acting on each $\bar{M}^{(k)}$ but measures per unit area of $M^{(k)}$, $\boldsymbol{t}^*$ and $\boldsymbol{h}^*$ be the external boundary force and moment resultant vectors prescribed along regular parts of the deformed boundary $\partial\bar{M}_f$ but measured per unit length of ∂M_f, while $\boldsymbol{f}_b^*$ be the external concentrated forces prescribed at each singular point $P_b \in \partial\bar{M}_f$. Then G_e in (1) takes the form [16]

$$G_e = \sum_k \iint_{M^{(k)}} \left(\boldsymbol{p}\cdot\delta\boldsymbol{u} + \boldsymbol{h}\cdot\delta\bar{\boldsymbol{n}}\right)da + \int_{\partial M_f}\left(\boldsymbol{t}^*\cdot\delta\boldsymbol{u} + \boldsymbol{h}^*\cdot\delta\bar{\boldsymbol{n}}\right)ds + \sum_{P_b\in\partial\bar{M}_f}\left(\boldsymbol{f}_b^*\cdot\delta\boldsymbol{u}_b\right), \tag{4}$$

where $\bar{\boldsymbol{n}}$ means the unit normal vector of $\bar{M}$.

It was proved in [16] that the most general form of G_Γ allowed within the non-linear theory of thin irregular shells is

$$G_\Gamma = \int_\Gamma \left(\boldsymbol{f}_\Gamma\cdot\delta\boldsymbol{u} + h_\Gamma\delta\varphi_\Gamma\right)ds + \sum_{P_i\in\bar{\Gamma}} \boldsymbol{f}_i\cdot\delta\boldsymbol{u}_i, \tag{5}$$

where $\boldsymbol{f}_\Gamma$ and h_Γ are the external force and couple-like loads distributed along regular parts of $\bar{\Gamma}$ but measured per unit length of Γ, $\boldsymbol{f}_i$ are the external concentrated forces applied at each singular point $P_i \in \bar{\Gamma}$, and φ_Γ is the scalar function describing the rotational deformation of the curve Γ.

When (3)-(5) are introduced into (1) one has to perform appropriate transformations. It should be taken into account that inside each smooth $M^{(k)}$ the kinematic fields $\boldsymbol{E},\bar{\boldsymbol{n}}$ are expressible through $\boldsymbol{u}$ and $\mathrm{Grad}_s\boldsymbol{u}$ while $\boldsymbol{K}$ depends also on $\mathrm{Grad}_s^2\boldsymbol{u}$, so that three components of $\boldsymbol{u}$ are the only independent field variables. But at regular points of $\partial M^{(k)}$ and hence along ∂M the field $\bar{\boldsymbol{n}}$ can be expressed in terms of $\boldsymbol{u}' \equiv \partial\boldsymbol{u}/ds$ and a scalar function $\varphi = \varphi(\boldsymbol{u},_\nu, \boldsymbol{u}')$ describing the rotational deformation of the shell lateral boundary surface, so that at edges of the shell base surface there are four independent field variables and $\delta\bar{\boldsymbol{n}} = \boldsymbol{L}\delta\boldsymbol{u}' + \boldsymbol{q}\delta\varphi$, see [14].

Let $\mathbf{u}$ denote symbolically the translation field $\boldsymbol{u}$ inside $M^{(k)} \in M$, the translation and rotation-like fields $(\boldsymbol{u},\varphi)$ and $(\boldsymbol{u}_\Gamma,\varphi_\Gamma)$ along regular parts of $\partial M^{(k)}$ and Γ, respectively, the translation $\boldsymbol{u}_i$ at each $P_i \in \Gamma$, and the translation $\boldsymbol{u}_b$ at each $P_b \in \partial M$. Then after complex transformations given in detail in [16] we obtain

$$
\begin{aligned}
G(\mathbf{u};\delta\mathbf{u}) = -\iint_{M\setminus\Gamma} & \left(\mathrm{Div}_s \boldsymbol{T} + \boldsymbol{l}\right)\cdot\delta\boldsymbol{u}\, da \\
& + \int_{\partial M_f} \left\{(\boldsymbol{p}_\nu - \boldsymbol{p}^* + \boldsymbol{k})\cdot\delta\boldsymbol{u} + (h - h^*)\delta\varphi\right\} ds + \sum_{P_b\in\partial\bar{M}_f} \left\{[\,\boldsymbol{f}\cdot\delta\boldsymbol{u}]_b - \boldsymbol{f}_b{}^*\cdot\delta\boldsymbol{u}_b\right\} \\
& + \int_{\partial M_d} \left\{(\boldsymbol{p}_\nu + \boldsymbol{k})\cdot\delta\boldsymbol{u} + h\delta\varphi\right\} ds + \sum_{P_b\in\partial\bar{M}_d} [\,\boldsymbol{f}\cdot\delta\boldsymbol{u}]_b \\
& + \int_{\Gamma} \left\{([\boldsymbol{p}_\nu + \boldsymbol{k}] - \boldsymbol{f}_\Gamma)\cdot\delta\boldsymbol{u}_\Gamma + ([h] - h_\Gamma)\delta\varphi_\Gamma\right\} ds + \sum_{P_i\in\bar{\Gamma}} \left\{[\,\boldsymbol{f}\,]_i - \boldsymbol{f}_i\right\}\cdot\delta\boldsymbol{u}_i = 0.
\end{aligned}
\tag{6}
$$

In (6), the compound tensor $\boldsymbol{T}$, vector $\boldsymbol{l}, \boldsymbol{p}_\nu, \boldsymbol{p}^*, \boldsymbol{k}$ and scalar h, h^* fields are defined in [2] through corresponding fields $\boldsymbol{N}, \boldsymbol{M}$ and $\boldsymbol{p}, \boldsymbol{h}, \boldsymbol{t}^*, \boldsymbol{h}^*$ described above, $[...]$ means the jump of $(...)$ along each regular part of Γ, $[...]_i$ is the jump of $(...)$ at each singular point of Γ, while $[...]_b$ means the jump of $(...)$ at each singular point of ∂M.

For any kinematically admissible virtual displacement $\delta\mathbf{u}$ the fields $\delta\boldsymbol{u}$ and $\delta\varphi$ identically vanish along ∂M_d, so that the third line of (6) identically vanish as well. Then the transformed PVW (6) requires the equilibrium equations, the static boundary and corner conditions as well as the corresponding jump conditions along Γ to be satisfied. In such formulation the kinematic relations, the material and junction characterisation by the constitutive equations as well as the kinematic boundary conditions should additionally be specified. The whole set of shell relations constitute the highly non-linear boundary value problem (BVP) in terms of translations as the only independent field variables.

3 Remarks on incremental-iterative solution of the BVP

The non-linear BVP of the irregular thin elasto-plastic shells with deformable junctions can effectively be solved by numerical methods applying some incremental-iterative solution procedure. The procedure is usually based on approximation of the non-linear BVP by series of linearised BVPs. For the Lagrangian non-linear theory of thin, regular elastic shells (without junctions) such solution procedure was worked out in [22], where the general structure of incremental shell equations and corresponding buckling shell equations were explicitly derived. However, in case of highly non-linear irregular elasto-plastic shell problems it is more efficient to apply the numerical incremental-iterative procedure directly to the incremental variational functional, not to the shell equations following from it.

Let us briefly recall some statements of [22] and extend them to shell problems with deformable junctions. Notice that components of the external loads $\boldsymbol{p}, \boldsymbol{h}$ in M, $\boldsymbol{t}^*, h^*, \boldsymbol{f}_b^*$ along ∂M_f and $\boldsymbol{f}_\Gamma, \boldsymbol{f}_i$ along Γ may be specified entirely independently, in general, by now 18 dimensionless parameters $\lambda_p \in \Lambda \subset \mathrm{R}^{18}$. Then the non-linear BVP for a thin irregular shell generated by (6) can be presented symbolically as $F(\mathbf{u},\lambda_p) = 0$, where the non-linear continuously differentiable operator F is defined on the product space $C(M,E^3)\times\mathrm{R}^{18}$ with values in the Banach space, where $C(M,E^3)$ is a set of all components of $\mathbf{u}$ and its surface gradients up to the 4th order.

In engineering applications all external loads are usually specified by a single common parameter $\lambda\in\Lambda\subset\mathrm{R}$. In this case the solution $\mathbf{u}(\lambda)$ of the BVP form a one-dimensional submanifold in $C(M,E^3)$ usually called the equilibrium path. $\mathbf{u}(\lambda)$ is called the weak solution if $G(\mathbf{u}(\lambda);\delta\mathbf{u}(\lambda))=0$ for all kinematically admissible $\delta\mathbf{u}(\lambda)$. For finding the weak solution $\mathbf{u}(\lambda)$ one usually applies the Newton-Kantorovich method [13] based on successive approximations to the exact solution at some λ_{m+1} through solving a series of linearised BVPs following from linearisation of $G(\mathbf{u}(\lambda);\delta\mathbf{u}(\lambda))=0$ about a λ_m close to λ_{m+1}.

Let us assume that smooth changes of λ generate locally regular weak solutions of the BVP. One can divide the equilibrium path $\mathbf{u}(\lambda)$ into a finite number of equilibrium states corresponding to increasingly ordered discrete values of the load parameter $\lambda_0,\lambda_1,...,\lambda_m,\lambda_{m+1},...\in\Lambda$. It is assumed that the weak solution $\mathbf{u}_m$ at the value λ_m is known, and the solution $\mathbf{u}_{m+1}$ at the next value λ_{m+1} is to be found from the known *i*-th approximation $\mathbf{u}_{m+1}^{(i)}$ which may not belong to the equilibrium path. To find the correction $\Delta\mathbf{u}_{m+1}^{(i+1)}$ allowing one to calculate the next approximation $\mathbf{u}_{m+1}^{(i+1)}=\mathbf{u}_{m+1}^{(i)}+\Delta\mathbf{u}_{m+1}^{(i+1)}$ to $\mathbf{u}_{m+1}$, one can linearise $G(\mathbf{u};\delta\mathbf{u})=0$ at the approximation $\mathbf{u}_{m+1}^{(i)}$ which leads to the following equation for $\Delta\mathbf{u}_{m+1}^{(i+1)}$ (see [13,24,18]):

$$G\left(\mathbf{u}_{m+1}^{(i)};\delta\mathbf{u}\right)+\Delta G\left(\mathbf{u}_{m+1}^{(i)};\Delta\mathbf{u}_{m+1}^{(i+1)},\delta\mathbf{u}\right)=0. \qquad (7)$$

The first term in (7) represents the value of G at the approximation $\mathbf{u}_{m+1}^{(i)}$. Since this approximation may not belong to the equilibrium path, the first term in (7) does not vanish, in general, and allows one to calculate the unbalanced force vector at the configuration corresponding to $\mathbf{u}_{m+1}^{(i)}$. The second term in (7) linear with regard to $\Delta\mathbf{u}_{m+1}^{(i+1)}$ allows one to calculate the tangent stiffness matrix at $\mathbf{u}_{m+1}^{(i)}$ of the non-linear BVP. If $\mathbf{u}_{m+1}$ corresponds to the regular solution point then the successive approximations $\mathbf{u}_{m+1}^{(i+1)}$ established by this method converge to $\mathbf{u}_{m+1}$ with velocity of geometrical progression, provided that the initial approximation $\mathbf{u}_{m+1}^{(0)}$ is sufficiently close to $\mathbf{u}_{m+1}$. Thus, efficiency of the Newton-Kantorovich method primarily depends on the choice of appropriate initial approximation $\mathbf{u}_{m+1}^{(0)}$.

The incremental shell relations following from (7) are valid for unrestricted translations, rotations, strains and/or bendings of the shell irregular base surface, arbitrary configuration-dependent external static loading, an arbitrary combination of work-conjugate boundary conditions, and arbitrary incremental constitutive relations of the shell and the junctions.

4 Constitutive elasto-plastic modelling in thin shells

Analysis in the elasto-plastic range of deformation of thin irregular shells with deformable junctions can be performed by the finite element method first proposed in [18], which has been extended here to account of deformability of the shell junctions. The shell is first divided into n layers and the plane stress state is assumed within each layer. The 3D incremental constitutive equations of each layer are described by the generalized elasto-plastic law of Prandtl-Reuss for small strains, with the associated flow rule and plasticity condition of Huber-Mises Hencky (HMH) with linear combination of isotropic and kinematic hardening.

In particular, in our constitutive shell model we apply the following relations (compare for example [34]):

1. Additive decomposition of differential increment of the Green strain tensor $\mathbf{e}$ into elastic and plastic parts,
$$d\mathbf{e} = d\mathbf{e}^E + d\mathbf{e}^P \,. \tag{8}$$
2. The overstress tensor,
$$\boldsymbol{\Sigma} = \mathbf{s} - \boldsymbol{\alpha}\,, \quad \boldsymbol{\Sigma}' = \mathbf{s}' - \boldsymbol{\alpha}\,, \quad \boldsymbol{\Sigma}' = \boldsymbol{\Sigma} - \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\Sigma})\mathbf{I}\,, \tag{9}$$
where $\mathbf{s}$ is the 2nd Piola-Kirchhoff stress tensor, $\mathbf{s}' = \mathbf{s} - \tfrac{1}{3}\,\mathrm{tr}(\mathbf{s})\mathbf{I}$ is its deviatoric part, $\boldsymbol{\alpha}$ is the corresponding back stress tensor, and $\mathbf{I}$ is the identity tensor of the 3D vector space.
3. The Huber-Mises-Hencky (HMH) yield condition,
$$f(\boldsymbol{\Sigma}, \bar{\varepsilon}^P) = \bar{\sigma} - \sigma_Y(\bar{\varepsilon}^P) = 0\,, \quad \bar{\sigma} = \sqrt{\tfrac{3}{2}\boldsymbol{\Sigma}' : \boldsymbol{\Sigma}'}\,, \quad \frac{\partial f}{\partial \boldsymbol{\Sigma}} \equiv \mathbf{r}\,, \tag{10}$$
where $\bar{\sigma}$ is the HMH effective stress and σ_Y is the yield stress in uniaxial tension.
4. The associated plastic flow rule and evolution equation,
$$\begin{aligned} &d\mathbf{e}^P = d\lambda\,\mathbf{r}\,, \quad d\lambda = \frac{\mathbf{r}:\mathbf{C}^E:d\mathbf{e}}{H' + \mathbf{r}:\mathbf{C}^E:\mathbf{r}}\,, \quad d\bar{\varepsilon}^P = \sqrt{\tfrac{2}{3}\,d\mathbf{e}^P : d\mathbf{e}^P}\,, \\ &d\boldsymbol{\alpha} = (1-\beta)H' d\mathbf{e}^P\,, \quad d\sigma_Y = \beta H' d\bar{\varepsilon}^P\,, \end{aligned} \tag{11}$$
where $\mathbf{C}^E$ is the 4th-order tensor of elastic moduli, $\bar{\varepsilon}^P$ is the accumulated effective plastic strain, H' is the strain hardening parameter, and $\beta \in [0,1]$ is the material parameter determining proportion between isotropic and kinematic hardening.
5. The incremental constitutive relation of the elasto-plastic continuum,
$$d\mathbf{s} = \mathbf{C}^{EP} : d\mathbf{e}\,, \tag{12}$$
where $\mathbf{C}^{EP}$ is the instantaneous tangent 4th-order tensor of the elasto-plastic material behaviour given by
$$\mathbf{C}^{EP} = \mathbf{C}^E - \frac{(\mathbf{C}^E:\mathbf{r}) \otimes (\mathbf{r}:\mathbf{C}^E)}{H' + \mathbf{r}:\mathbf{C}^E:\mathbf{r}}\,. \tag{13}$$

In this approach, as the hardening function we can also take the multi-segment approximation of experimental curves following from material tests in tension, if necessary. The stress increments corresponding to the strain increments are calculated from velocity relations using the Euler method of forward integration with correction following from the plasticity condition. Then the incremental constitutive equations for the shell stress resultants and stress couples are established by direct through-the-thickness integration throughout all layers of 3D relations mentioned above. All matrix relations for the finite element are calculated numerically using 3-point Gauss integration within the element, and up to $n = 10$ integration points across the shell thickness are applied.

5 Example: Axisymmetric casing with deformable junctions undergoing elasto-plastic deformation

The axisymmetric casing of measuring device being a part of a pressure installation consists of three regular thin shells of revolution: the circular cylindrical part of thickness h_c, length H and diameter D, and two toroidal parts of thickness h_s, inner boundary diameter d_z and radius r. These dimensions are related by $D = d_z + 2r$, see Fig. 1.

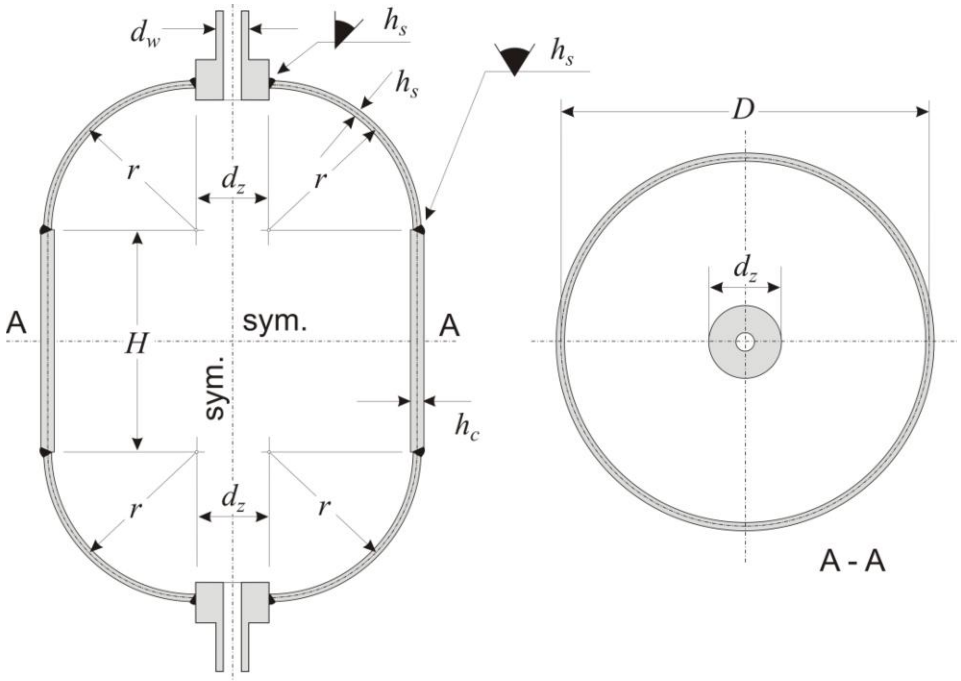

Figure 1. The axisymmetric casing: geometry

The toroidal parts are connected with the cylindrical one by welding, while at the upper and lower ends the toroidal parts are connected with rigid parts by welding or screw joints. Thus, in this example we have different technological inaccuracies at the junctions associated with welding (or screw joints) and with change of thickness.

The force $P_{(a)}$ acting at the inner and lower toroidal boundaries, see Fig. 2, comes from pressure difference applied to rigid parts of the casing and is calculated according to

$$P_{(a)} = \frac{1}{4} q \pi \left(d_z^2 - d_w^2 \right). \tag{14}$$

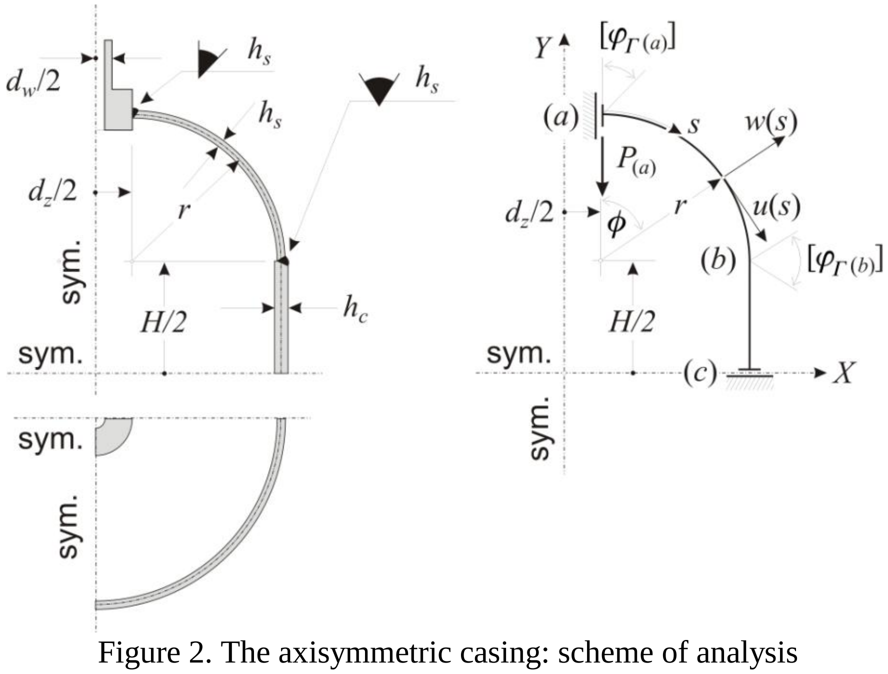

Figure 2. The axisymmetric casing: scheme of analysis

In the numerical analysis of this example we use the program MINIMOD [7] with the axisymmetric two-node RING element based on the theory of thin shells with finite rotations proposed in [21,23]. In this element two translation components u, w are approximated using the Hermite interpolation with C^1 interelement continuity of the form

$$\begin{Bmatrix} \tilde{u} \\ \tilde{w} \end{Bmatrix} = \sum_{k=1}^{2} \left(H_k^0 \begin{Bmatrix} u_k \\ w_k \end{Bmatrix} + H_{sk}^1 \begin{Bmatrix} u,_{sk} \\ w,_{sk} \end{Bmatrix} \right), \tag{15}$$

where H_k^0 and H_{sk}^1 are shape functions in the Hermite interpolation.

In the analysis the following numerical data have been used: $h_c = 2\,\text{mm}$, $h_s = 1\,\text{mm}$, $H = 50\,\text{mm}$, $D = 100\,\text{mm}$, $d_z = 10\,\text{mm}$, $d_w = 5\,\text{mm}$, $r = 45\,\text{mm}$. Within the elastic range of deformation we take $E = 210\,\text{GPa}$ and $\nu = 0{,}3$. The plastic range of deformation is characterized by the initial plasticity limit $\sigma_0 = \sigma_Y(0) = 450\,\text{MPa}$, the mixed isotropic-kinematic hardening is described by parameter $\beta = 1/2$ and the tangent modulus is taken as $E_T = 0{,}001 \times E$. The linear constitutive relation $h_\Gamma = c([\varphi_\Gamma])$ governs deformability of the junctions. In the analysis we discuss only two extreme cases $c = 0$ and $c = \infty$ to better read off the results.

The numerical results are presented in Fig. 3-5.

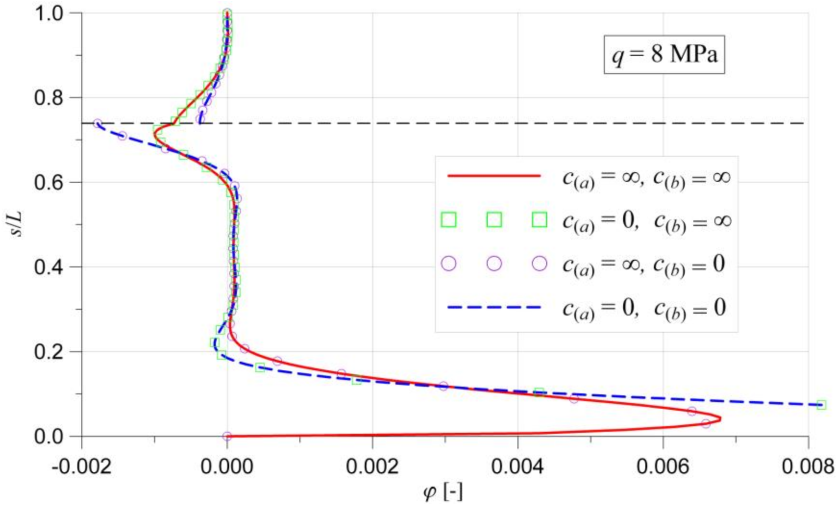

Figure 3. The angle of rotation along the casing for $q = 8$ MPa

In Fig. 3 we show how the angle of rotation φ changes along the dimensionless parameter s/L. The length L on the vertical axis of Fig. 3 is defined as $L = 1/2(H + \pi r)$, so that $s/L \in (0,1)$. Due to small values of translations $w(s)$ and $u(s)$ the quantity $\varphi = w,_s + \phi,_s u$ approximately describes the through-the-thickness average infinitesimal angle of rotation of material elements during the elasto-plastic deformation indeed. Notice that when both junctions at points (a) and (b) are stiff, i.e. $c_{(a)} = c_{(b)} = \infty$ (the continuous curve), the values of φ at the end points (a) and (c) become zero and the curve has no jumps. For both simply supported junctions, i.e. $c_{(a)} = c_{(b)} = 0$ (the dashed curve), the values of φ at these points exhibit jumps confirming discontinuities of the rotation at the junctions. In the mixed cases, when either $c_{(a)} = 0$, $c_{(b)} = \infty$ (the curve with squares) or $c_{(a)} = \infty$, $c_{(b)} = 0$ (the curve with circles), both curves have only one jump at the simply supported junction, and this effect is pronounced only locally.

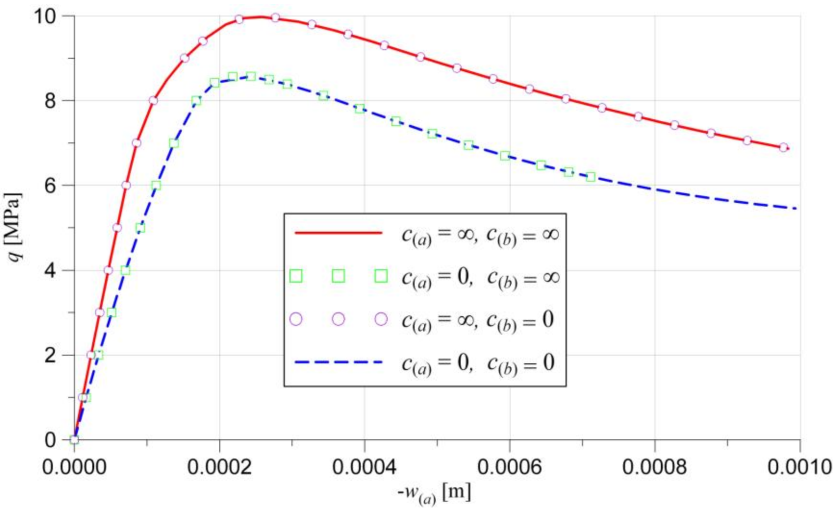

Figure 4. Vertical translation of the point (*a*) as function of pressure

Fig. 4 indicates how the vertical translation $w_{(a)}$ depends on the external pressure applied to the casing. With grooving q the $w_{(a)}$ grows initially almost linearly up to about 6 MPa. Above that value the plastic material behaviour makes the measurements of pressure less and less accurate up to about 8 MPa. Above the latter value the graphs $w_{(a)} = f(q)$ become non-linear with clearly pronounced limit points for q above which the devise becomes damaged. The maximal value of the limit point corresponds to $c_{(a)} = \infty$, and its minimum value to $c_{(a)} = 0$. The stiffness of $c_{(b)}$ has no influence on these results.

Finally, on Fig. 5 the distribution of the bending couple M^1 along the casing for $q = 8$ MPa is presented. It is seen that for $c_{(a)} = c_{(b)} = 0$ there are zero values of the couple at both junctions, and the stiff characteristic of either junction has only local effect on the overall distribution of couples.

Other intermediate values of the junction stiffness $0 < \left(c_{(a)}, c_{(b)}\right) < \infty$ would lead to similar graphs lying between those indicated in Fig. 3-5.

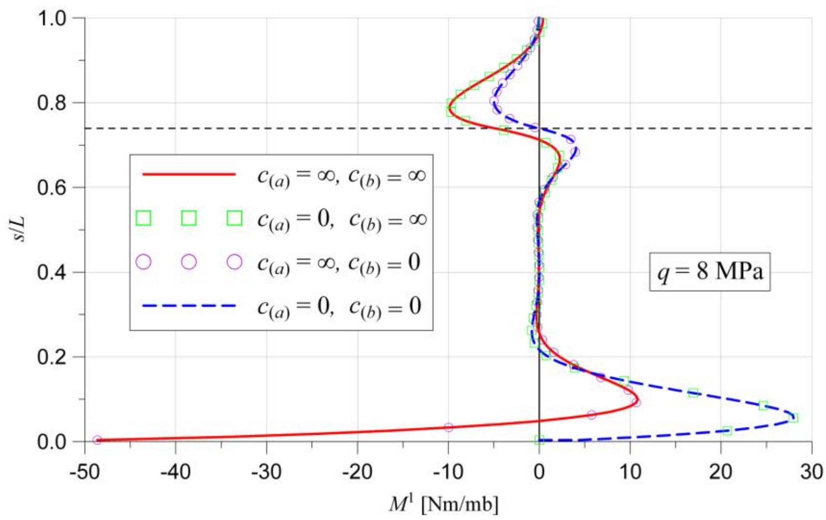

Figure 5. Bending couple along the casing for $q = 8$ MPa

6 Conclusions

For the irregular thin shell structures we have proposed a modified form of the principle of virtual work in which stiffness characteristics of deformable junctions are explicitly taken into account. The PVW has then been discretized by C^1 finite elements and the usual incremental-iterative procedure has been applied to find the solution of the non-linear BVP. Numerical results of axisymmetric behaviour of the casing under external pressure in the elasto-plastic range of deformation indicates, among others, that deformability characteristics of the junctions influence the results only locally.

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