Chapter 16
On the Nonlinear Theory of Two-Phase Shells

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Abstract We discuss the nonlinear theory of shells made of material undergoing phase transitions (PT). The interest to such thin-walled structures is motivated by applications of thin films made of martensitic materials and needs of modeling biological membranes. Here we present the resultant, two-dimensional thermodynamics of non-linear theory of shells undergoing PT. The global and local formulations of the balances of momentum, moment of momentum, energy and entropy are given. Two temperature fields on the shell base surface are introduced: the referential mean temperature and its deviation, as well as two corresponding dual fields: the referential entropy and its deviation. Additional surface heat flux and the extra heat flux vector fields appear as a result of through-the-thickness integration procedure. Within the framework of the resultant shell thermodynamics we derive the continuity conditions along the curvilinear phase interface which separates two material phases. These conditions allow us to formulate the kinetic equation describing the quasistatic motion of the interface relative to the shell base surface. The kinetic equation is expressed by the jump of the Eshelby tensor across the phase interface. In the case of thermodynamic equilibrium the variational statement of the static problem of two-phase shell is presented.

Keywords Non-linear shell · Shell thermodynamics · Phase transition · Cosserat shell · Micropolar shell · Kinetic equation · Singular curve

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16.1 Introduction

The interest in thin-walled structures undergoing PT grows recently from their perspective applications in engineering. As examples of such structures martensitic films and biological membranes can be considered. The stress- and temperature-induced PT are widely observed in thin structures made of superelastic shape memory alloys (SMA) and shape memory polymers, such as NiTi, NiMnGa, AgCd, AuCd, CuAlNi, polyurethane, etc, which are used in various microelectromechanical systems (MEMS). Thin plates, strips, and tubes made of SMA are used as working elements of such MEMS as micropumps, sensors, actuators, microengines etc., see e.g. [8, 48, 53] and books [5, 32, 38].

Experiments on shape memory alloys and other materials undergoing PT are often performed with thin-walled samples such as thin strips, rectangular plates or thin tubes, see [25–28, 33, 41, 52, 56] among others. Other examples of PT in thin-walled structures are tents and tunnels appearing in martensitic thin films during PT, which were discovered and investigated in [6, 8, 24, 29, 47, 48].

The major known theories of PT in deformable solids are related to three-dimensional (3D) thermoelasticity, see the books [1, 4, 5, 22] and references given therein. The first two-dimensional (2D) mechanical model of PT in thin films was proposed in [7, 30, 47], see also [5, 38]. The model consists of the Cosserat membrane with one director, but without taking into account the bending stiffness of the membrane. Alternative membrane models of PT with applications to biomembrane modelling were proposed in [2, 10, 12].

The non-linear resultant equilibrium conditions of elastic shells undergoing PT of martensitic type were formulated by Eremeyev and Pietraszkiewicz [13] within the resultant dynamically exact and kinematically unique theory of shells presented in [11, 18, 34]. These conditions were extended in [44] taking into account the line tension energy of the interface. By analogy to the 3D case, the two-phase shell was regarded in [13,44] as the Cosserat surface consisting of two material phases divided by a sufficiently smooth surface singular curve (phase interface). Existence of such a curve was confirmed by several experiments on thin-walled plates, strips, and tubes, see e.g. [25–28,33,41,52,56]. These experiments demonstrate how the macroscopic domain of the new phase forms, show its further evolution during loading and annihilation after unloading. In the case of plates and strips the new phase forms often as a few bands across the strip. In the case of tubes the new phase may appear as helical or cylindrical bands which width and shape depend on acting loads. The phase boundary between “old” and “new” phases in many cases can be interpreted as a curvilinear, coherent, sharp phase interface. The quasistatic behaviour of two-phase shells has recently been analyzed in [15–17].

In this paper we discuss the resultant, two-dimensional thermomechanics of shells undergoing diffusionless, displacive phase transitions of martensitic type of the shell material. In particular, we formulate the corresponding boundary-value problem (BVP). The main attention is paid to formulation of the compatibility conditions across the phase interface and to derivation of the kinetic equation describing propagation of the phase interface during loading and unloading.

16.2 Kinematics of 6-Parametric Theory of Shells

The kinematics of general resultant theory of shells coincides with the kinematics of 2D Cosserat continuum, see [11, 18, 34] for details. In the undeformed placement the shell is represented by the base surface $\mathcal{M}$ with the position vector $\boldsymbol{x}(\theta^\alpha)$ and the unit normal vector $\boldsymbol{n}(\theta^\alpha)$, where $\{\theta^\alpha\}$, $\alpha = 1,2$, are the surface curvilinear coordinates. In the deformed placement the shell is represented by the surface $\mathcal{N} = \chi(\mathcal{M})$ with the position vector $\boldsymbol{y} = \chi(\boldsymbol{x})$ and with the attached three directors $(\boldsymbol{d}_\alpha, \boldsymbol{d})$. The deformation of the shell is described by the relations

$$\boldsymbol{y}(\boldsymbol{x},t) = \chi(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x},t), \quad \boldsymbol{d}_\alpha(\boldsymbol{x},t) = \boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{x}_{,\alpha}, \quad \boldsymbol{d}(\boldsymbol{x},t) = \boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x}), \tag{16.1}$$

where t is a time-like scalar parameter, χ the deformation function, $\boldsymbol{u} \in E$ the translation vector of $\mathcal{M}$, and $\boldsymbol{Q} \in SO(3)$ the proper orthogonal tensor representing the work-averaged gross rotation of the shell cross sections from their undeformed shapes described by $(\boldsymbol{x}_{,\alpha}, \boldsymbol{n})$, where $(\ldots)_{,\alpha}$ denotes partial differentiation with respect to θ^α. Then $\boldsymbol{v} \equiv \dot{\boldsymbol{u}}$ is the translation velocity and $\boldsymbol{\omega} \equiv \mathrm{ax}(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T)$ the angular velocity vectors, where $\mathrm{ax}(\ldots)$ is the axial vector associated with the skew tensor $(\ldots)$, and $\dot{(\ldots)}$ denotes the derivative with respect to t.

Within the framework of 6-parametric theory of shells considered here, the following two strain measures corresponding to the deformations (16.1) are introduced, see [11, 13, 14, 43]:

$$\boldsymbol{E} = \boldsymbol{\varepsilon}_\alpha \otimes \boldsymbol{a}^\alpha, \quad \boldsymbol{K} = \boldsymbol{\varkappa}_\alpha \otimes \boldsymbol{a}^\alpha, \quad \boldsymbol{\varepsilon}_\alpha = \boldsymbol{y}_{,\alpha} - \boldsymbol{d}_\alpha, \quad \boldsymbol{\varkappa}_\alpha = \frac{1}{2}\boldsymbol{d}^i \times \boldsymbol{Q}_{,\alpha}\boldsymbol{Q}^T\boldsymbol{d}_i, \tag{16.2}$$

where $(\boldsymbol{a}^\alpha, \boldsymbol{n})$ and $(\boldsymbol{d}^i)$, $i = 1,2,3$, are bases reciprocal to the base $(\boldsymbol{x}_{,\alpha}, \boldsymbol{n})$ and the base $(\boldsymbol{d}_\alpha, \boldsymbol{d})$, respectively.

We assume that in the deformed placement the shell consists of different material phases occupying different complementary subregions separated by the curvilinear phase interface $\mathcal{D} \in \mathcal{N}$. For a piecewise differentiable mapping χ we can introduce on $\mathcal{M}$ a singular image curve $C = \chi^{-1}(\mathcal{D})$ with the position vector $\boldsymbol{x}_C$. We call a priori unknown curves $\mathcal{D}$ and C the phase interfaces in the deformed and reference placements, respectively. Let us note that $\boldsymbol{x}_C$ and $\boldsymbol{y}_\mathcal{D}$ are kinematically independent on $\boldsymbol{u}$ and $\boldsymbol{Q}$. This means that $\mathcal{D}$ and C are non-material curves, in general. For the description of motion of the surface curve C on $\mathcal{M}$ we introduce the phase interface velocity $V \equiv \dot{\boldsymbol{x}}_C \cdot \boldsymbol{\nu}$, where $\boldsymbol{\nu} \in T_x\mathcal{M}$ is the unit external normal vector to C, and $\boldsymbol{\nu} \cdot \boldsymbol{n} = 0$.

Hence, $\boldsymbol{y}$ (or $\boldsymbol{u}$), $\boldsymbol{Q}$, and $\boldsymbol{x}_C$ constitute the basic kinematic unknown variables in the theory of shells undergoing PT.

16.3 Integral Balance Equations

The resultant 2D equations of the general non-linear theory of shells can be derived exactly by direct through-the-thickness integration of 3D balance laws of linear and angular momentum as well as of the energy balance and the entropy inequality of continuum mechanics, see [11, 13, 17, 34]. In quasi-static problems discussed here the global equilibrium conditions require the total force and total torque of all loads acting upon any part $\mathcal{P} \subset \mathcal{M}\backslash C$ to vanish

$$\mathfrak{F} = \mathbf{0}, \quad \mathfrak{M} = \mathbf{0}, \tag{16.3}$$

where

$$\mathfrak{F} \equiv \iint\limits_{\mathcal{P}} \boldsymbol{f}\, da + \int\limits_{\partial\mathcal{P}\backslash\partial\mathcal{M}_f} \boldsymbol{n}_\nu\, ds + \int\limits_{\partial\mathcal{P}\cap\partial\mathcal{M}_f} \boldsymbol{n}^*\, ds,$$

$$\mathfrak{M} \equiv \iint\limits_{\mathcal{P}} (\boldsymbol{c} + \boldsymbol{y}\times\boldsymbol{f})\, da + \int\limits_{\partial\mathcal{P}\backslash\partial\mathcal{M}_f} (\boldsymbol{m}_\nu + \boldsymbol{y}\times\boldsymbol{n}_\nu)\, ds + \int\limits_{\partial\mathcal{P}\cap\partial\mathcal{M}_f} (\boldsymbol{m}^* + \boldsymbol{y}\times\boldsymbol{n}^*)\, ds.$$

Here $\boldsymbol{f}$ and $\boldsymbol{c}$ are the resultant surface force and couple vector fields acting on $\mathcal{N}\backslash\mathcal{D}$, but measured per unit area of $\mathcal{M}\backslash C$. Similarly, $\boldsymbol{n}_\nu$ and $\boldsymbol{m}_\nu$ are the internal contact stress and couple resultant vectors defined at an arbitrary edge $\partial\mathcal{R}$ of $\mathcal{R} = \chi(\mathcal{P})$, while $\boldsymbol{n}^*$ and $\boldsymbol{m}^*$ are the external boundary resultant force and couple vectors applied along the part $\partial\mathcal{N}_f$ of $\mathcal{N} = \chi(\mathcal{M})$, respectively. The latter four vectors are measured per unit length of the corresponding undeformed edges $\partial\mathcal{P}$ and $\partial\mathcal{M}_f$, respectively.

According to the Cauchy postulate, the contact vectors $\boldsymbol{n}_\nu$ and $\boldsymbol{m}_\nu$ can be represented through the respective internal surface stress and couple resultant tensors $\boldsymbol{N}$ and $\boldsymbol{M}$ by

$$\boldsymbol{n}_\nu = \boldsymbol{N}\boldsymbol{\nu}, \quad \boldsymbol{m}_\nu = \boldsymbol{M}\boldsymbol{\nu}. \tag{16.4}$$

The tensors $\boldsymbol{N}$, $\boldsymbol{M} \in E \otimes T_x\mathcal{M}$ defined on $\mathcal{M}\backslash C$ are the resultant surface stress measures of the 1st Piola–Kirchhoff type, respectively.

In the literature various descriptions of shell thermodynamics are known, see e.g. [15, 18, 20, 21, 35, 39, 40, 45, 46, 49–51, 57], where various sets of surface fields responsible for temperature were used and several formulations of the first and second laws of thermodynamics for shells were discussed.

The resultant local thermomechanic energy balance and the entropy inequality for the shell can also be derived by direct through-the-thickness integration of the global 3D thermomechanic balance of energy and the entropy inequality, see [35, 49–51].

However, in the construction of 2D thermodynamic relations for shells one cannot simply transfer the notions of temperature, entropy and energy from the 3D case to their averages defined on $\mathcal{M}$. For example, in the 3D shell-like body it is quite natural to associate different temperatures with its lower, upper and lateral boundary surfaces. In the resultant shell model this leads to appearance of several 2D temperature fields defined in the same point of the base surface. Each such field may require

its own heat conduction law. As a result, the 2D entropy inequality for shells should also be appropriately modified.

Similarly, the 3D stress power density of the shell-like body takes into account also a part of stress power not expressible in terms of 2D stress resultants and stress couples, see [43], Sect. 7. In the 2D energy balance equation this may require to introduce additional surface sources of energy with their own constitutive relations, see discussion in [35, 42].

The referential form of energy balance *(The 1st Law of thermomechanics)* of an arbitrary part $\mathcal{P}$ of the shell base surface $\mathcal{M}\backslash C$ can be described in analogy to the 3D energy balance, see [54, 55], by the resultant quantities [35] as

$$\dot{\mathfrak{K}} + \dot{\mathfrak{E}} = \mathfrak{A} + \mathfrak{Q}, \tag{16.5}$$

where $\mathfrak{K}$ is the resultant kinetic energy, $\mathfrak{E}$ the resultant internal energy, $\mathfrak{A}$ the resultant mechanical power, and $\mathfrak{Q}$ is the resultant heating. For the quasistatic process discussed here $\mathfrak{K} = 0$, while $\mathfrak{E}$, $\mathfrak{A}$, and $\mathfrak{Q}$ can be represented by

$$\mathfrak{E} \equiv \iint\limits_{\mathcal{P}} \rho\epsilon\, da, \quad \mathfrak{A} \equiv \iint\limits_{\mathcal{P}} (\boldsymbol{f}\cdot\boldsymbol{v} + \boldsymbol{c}\cdot\boldsymbol{\omega})\, da + \int\limits_{\partial\mathcal{P}\backslash\partial\mathcal{M}_f} (\boldsymbol{n}_\nu\cdot\boldsymbol{v} + \boldsymbol{m}_\nu\cdot\boldsymbol{\omega})\, ds$$
$$+ \int\limits_{\partial\mathcal{P}\cap\partial\mathcal{M}_f} (\boldsymbol{n}^*\cdot\boldsymbol{v} + \boldsymbol{m}^*\cdot\boldsymbol{\omega})\, ds,$$
$$\mathfrak{Q} \equiv \iint\limits_{\mathcal{P}} \rho r\, da + \int\limits_{\partial\mathcal{P}\backslash\partial\mathcal{M}_h} q_\nu\, ds + \int\limits_{\partial\mathcal{P}\cap\partial\mathcal{M}_h} q^*\, ds,$$

where ρ is the resultant surface mass density in undeformed placement, ε the internal resultant surface strain energy density per unit undeformed surface mass, and r the internal surface heat supply minus heat fluxes through the upper and lower shell faces, all per unit mass of $\mathcal{M}$, q_ν and q^* are the surface heat fluxes through $\partial\mathcal{P}$ and $\partial\mathcal{M}_h$, respectively. The contact heat flux q_ν can be represented through the surface heat flux vector $\boldsymbol{q}$ by the formula

$$q_\nu = \boldsymbol{q}\cdot\boldsymbol{\nu}.$$

The referential form of entropy inequality *(The 2nd Law of thermomechanics)* of an arbitrary part $\mathcal{P}$ of the shell base surface $\mathcal{M}\backslash C$ follows from the Clausius-Duhem inequality [54, 55],

$$\dot{\mathfrak{H}} \geq \mathfrak{J}, \tag{16.6}$$

where in our case $\mathfrak{H}$ is the resultant shell entropy and $\mathfrak{J}$ the resultant entropy supply. For any part $\mathcal{P} \subset \mathcal{M}\backslash C$ these fields are defined as follows:

$$\mathfrak{H} \equiv \iint\limits_{\mathcal{P}} \rho\eta\, da, \quad \mathfrak{J} \equiv \iint\limits_{\mathcal{P}} \rho j\, da + \int\limits_{\partial\mathcal{P}\backslash\partial\mathcal{M}_h} j_\nu\, ds + \int\limits_{\partial\mathcal{P}\cap\partial\mathcal{M}_h} j^*\, ds,$$

where η is the resultant internal entropy density, j the resultant entropy supply minus entropy fluxes through the upper and lower shell faces, both per unit undeformed surface mass, and j_ν and j^* are the resultant entropy fluxes through the internal $\partial\mathcal{P}$ and external $\partial\mathcal{M}_h$ boundary contours, respectively. The field j_ν can be expressed through the referential entropy flux vector $\boldsymbol{j} \in T_x\mathcal{M}$ according to

$$j_\nu = \boldsymbol{j}\cdot\boldsymbol{\nu}.$$

The relations between the resultant quantities and their 3D counterparts can be derived by use of the through-the-thickness integration procedure applied to the 3D balance equations [17]. After [39] we introduce the mean referential temperature $\theta(\boldsymbol{x},t) > 0$ and the temperature deviation $\varphi(\boldsymbol{x},t)$ by

$$\frac{1}{\theta} = \frac{1}{2}\left(\frac{1}{\Theta_+} + \frac{1}{\Theta_-}\right), \quad \varphi = \frac{1}{h}\left(\frac{1}{\Theta_-} - \frac{1}{\Theta_+}\right), \tag{16.7}$$

where $\Theta_\pm > 0$ are temperatures of the upper and lower shell faces $\mathcal{M}^\pm$ taken to be equal to those prevailing in the adjoining external media, and h is the shell thickness.

Unlike in the 3D entropy balance [54, 55], the resultant entropy supply j and the resultant entropy flux $\boldsymbol{j}$ take now the extended form [17],

$$j = \frac{1}{\theta}r - \varphi s, \quad \boldsymbol{j} = \frac{1}{\theta}\boldsymbol{q} - \varphi\boldsymbol{s}, \tag{16.8}$$

where s is the resultant extra heat supply and $\boldsymbol{s}$ is the resultant extra heat flux vector.

16.4 Local Shell Equations and Constitutive Relations

From the integral equilibrium equations (16.3), the energy balance equation (16.5) and the entropy inequality (16.6), after appropriate transformations follow the local Lagrangian equilibrium equations and the static boundary conditions

$$\mathrm{Div}\boldsymbol{N} + \boldsymbol{f} = \boldsymbol{0}, \quad \mathrm{Div}\boldsymbol{M} + \mathrm{ax}\left(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T\right) + \boldsymbol{c} = \boldsymbol{0} \quad \text{in } \mathcal{M}\backslash C, \tag{16.9}$$

$$\boldsymbol{N}\boldsymbol{\nu} - \boldsymbol{n}^* = \boldsymbol{0}, \quad \boldsymbol{M}\boldsymbol{\nu} - \boldsymbol{m}^* = \boldsymbol{0} \quad \text{along } \partial\mathcal{M}_f,$$

the local resultant thermomechanic balances of energy in the referential description

$$\rho\dot{\varepsilon} = \rho r - \mathrm{Div}\,\boldsymbol{q} + \boldsymbol{N}\bullet\boldsymbol{E}^\circ + \boldsymbol{M}\bullet\boldsymbol{K}^\circ \quad \text{in } \mathcal{M}\backslash C, \tag{16.10}$$

$$\boldsymbol{q}\cdot\boldsymbol{\nu} - q^* = 0 \quad \text{along } \partial\mathcal{M}_h,$$

and the local resultant entropy inequalities

$$\rho\dot{\eta}-\rho\left(\frac{r}{\theta}-\varphi s\right)+\frac{1}{\theta}\operatorname{Div}\boldsymbol{q}-\varphi\operatorname{Div}\boldsymbol{s}+\boldsymbol{h}\cdot\boldsymbol{s}-\frac{1}{\theta^2}\boldsymbol{q}\cdot\boldsymbol{g}\geq 0\quad\text{in}\quad\mathcal{M}\backslash C,\tag{16.11}$$

$$\frac{q^*}{\theta^*}-\varphi^* s^*-\left(\frac{q_\nu}{\theta}-\varphi s_\nu\right)\geq 0\quad\text{along}\quad\partial\mathcal{M}_h,$$

$$\boldsymbol{g}=\operatorname{Grad}\theta,\quad\boldsymbol{h}=\operatorname{Grad}\varphi,\qquad\boldsymbol{g},\boldsymbol{h}\in T_x\mathcal{M},$$

of the non-linear theory of shells. Here $\boldsymbol{F}\equiv\operatorname{Grad}\boldsymbol{y}=\boldsymbol{y}_{,\alpha}\otimes\boldsymbol{a}^\alpha$ is the surface deformation gradient, $\boldsymbol{F}\in E\otimes T_x\mathcal{M}$, $\operatorname{Div}\boldsymbol{N}\equiv\boldsymbol{N}_{,\alpha}\cdot\boldsymbol{a}^\alpha$ means the surface divergence of $\boldsymbol{N}$, $(\cdot)^\circ\equiv\boldsymbol{Q}\frac{d}{dt}[\boldsymbol{Q}^T(\cdot)]$ is the co-rotational time derivative, and the scalar product of two tensors $\boldsymbol{A},\boldsymbol{B}\in E\otimes T_x\mathcal{M}$ is defined by $\boldsymbol{A}\bullet\boldsymbol{B}\equiv\operatorname{tr}(\boldsymbol{A}^T\boldsymbol{B})$.

The fields $\boldsymbol{u},\boldsymbol{Q},\theta,\varphi$ constitute the basic thermo-kinematic independent variables of the shell boundary value problem in $\mathcal{M}\backslash C$, while the fields $\boldsymbol{N},\boldsymbol{M},\varepsilon,\eta,\chi,\boldsymbol{q}$, and $\boldsymbol{s}$ have to be specified by the constitutive equations.

Here, as an example we discuss the constitutive equations for thermoelastic shells which take the form [17],

$$\begin{gathered}\psi\equiv\varepsilon-\theta\eta-\varphi\chi=\psi(\boldsymbol{E},\boldsymbol{K},\theta,\varphi),\\ \boldsymbol{N}=\rho\psi_{,\boldsymbol{E}},\quad\boldsymbol{M}=\rho\psi_{,\boldsymbol{K}},\quad\eta=-\psi_{,\theta},\quad\chi=-\psi_{,\varphi},\\ \boldsymbol{q}=\boldsymbol{q}(\boldsymbol{E},\boldsymbol{K},\theta,\boldsymbol{g},\varphi,\boldsymbol{h}),\quad\boldsymbol{s}=\boldsymbol{s}(\boldsymbol{E},\boldsymbol{K},\theta,\boldsymbol{g},\varphi,\boldsymbol{h}),\end{gathered}\tag{16.12}$$

where we have introduced the surface free energy density ψ.

For thermoelastic shells the local energy balance equation (16.10) reduces to the form

$$\rho(\theta\dot{\eta}+\varphi\dot{\chi})=\rho r-\operatorname{Div}\boldsymbol{q},\tag{16.13}$$

while the local entropy inequality (16.11) results in the equation

$$-\rho\dot{\chi}+\rho\theta s-\theta\operatorname{Div}\boldsymbol{s}=c\varphi,\quad c\geq 0,\tag{16.14}$$

where the new constitutive function c is introduced, and the reduced dissipation inequality becomes

$$-\frac{1}{\theta}\boldsymbol{g}\cdot\boldsymbol{q}-\theta\boldsymbol{h}\cdot\boldsymbol{s}\geq 0.\tag{16.15}$$

Both relations (16.13) and (16.14) play the role of thermoconductivity equations in the theory of thermoelastic shells. The two equations are necessary to determine two fields: the surface mean temperature θ and the surface temperature deviation φ. When $\boldsymbol{s}=\boldsymbol{0}$, the equation (16.14) contains as the special case the equation for temperature deviation established for thermoelastic beams in [50].

The simplest cases of the constitutive equations for $\boldsymbol{q}$ and $\boldsymbol{s}$ satisfying (16.15) may be taken similar to the referential Fourier law in 3D continuum mechanics:

$$\boldsymbol{q}=-c_{\|}\boldsymbol{g},\quad\boldsymbol{s}=-c_{\perp}\boldsymbol{h},\tag{16.16}$$

where $c_{\parallel}$ is the positive heat conductivity of the shell in tangential direction and $c_{\perp}$ is the positive heat conductivity of the shell in the transverse normal direction.

16.5 Continuity Conditions Along Phase Interface and Kinetic Equation

Since all fields defined on $\mathcal{M}$ can be discontinuous across C, the phase interface C can be considered as a surface non-material singular curve. In particular, the curvilinear phase interfaces in shells can be either coherent or incoherent in rotations, see [13]. For the *coherent interface* both fields $\boldsymbol{y}$ (or $\boldsymbol{u}$) and $\boldsymbol{Q}$ are supposed to be continuous at C and the kinematic compatibility conditions along C become

$$[\![\boldsymbol{v}]\!] + V[\![\boldsymbol{F}\boldsymbol{\nu}]\!] = \boldsymbol{0}, \tag{16.17}$$

$$[\![\boldsymbol{\omega}]\!] + V[\![\boldsymbol{K}\boldsymbol{\nu}]\!] = \boldsymbol{0}, \tag{16.18}$$

where the expression $[\![\ldots]\!] = (\ldots)_B - (\ldots)_A$ means the jump at C.

The phase interface is called *incoherent in rotations* if only $\boldsymbol{y}$ (or $\boldsymbol{u}$) is continuous at C but $\boldsymbol{Q}$ may be discontinous. In this case the condition (16.17) is still satisfied, but (16.18) may be violated, see [13].

Assuming $[\![\boldsymbol{y}]\!] = \boldsymbol{0}$ along C, from (16.3) we obtain the local Lagrangian dynamic compatibility conditions [11],

$$[\![\boldsymbol{N}\boldsymbol{\nu}]\!] = \boldsymbol{0}, \quad [\![\boldsymbol{M}\boldsymbol{\nu}]\!] = \boldsymbol{0}, \tag{16.19}$$

which are just the local balances of forces and couples at C in the case of quasistatic deformations.

Additionally, we assume that the surface temperature field θ and its deviation φ are continuous on the whole M, that is

$$[\![\theta]\!] = 0, \quad [\![\varphi]\!] = 0 \quad \text{along } C. \tag{16.20}$$

The local energy balance and the entropy inequality along C corresponding to (16.5) and (16.6) are [17],

$$V[\![\rho\varepsilon]\!] + [\![\boldsymbol{N}\boldsymbol{\nu}\cdot\boldsymbol{v}]\!] + [\![\boldsymbol{M}\boldsymbol{\nu}\cdot\boldsymbol{\omega}]\!] - [\![\boldsymbol{q}\cdot\boldsymbol{\nu}]\!] = 0, \tag{16.21}$$

$$V[\![\rho\eta]\!] - \left[\!\!\left[\frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{\nu}\right]\!\!\right] + [\![\varphi\boldsymbol{s}\cdot\boldsymbol{\nu}]\!] \equiv \delta_C^2 \geq 0\,, \tag{16.22}$$

where $\delta_C^2 \geq 0$ denotes the surface entropy production along C.

The second thermoconductivity equation (16.14) leads to the relation along C,

$$V\frac{1}{\theta}[\![\rho\chi]\!] - [\![\boldsymbol{s}\cdot\boldsymbol{\nu}]\!] = 0\,. \tag{16.23}$$

From (16.17)–(16.23) we obtain the compatibility condition in the form

$$\theta\delta_C^2 = -V\left\{[[\rho\psi]] - \boldsymbol{\nu}\cdot\boldsymbol{N}^T[[\boldsymbol{F}\boldsymbol{\nu}]] - \boldsymbol{\nu}\cdot\boldsymbol{M}^T[[\boldsymbol{K}\boldsymbol{\nu}]]\right\} \quad \text{along } C \tag{16.24}$$

for the coherent phase interface, and

$$\theta\delta_C^2 = -V\left\{[[\rho\psi]] - \boldsymbol{\nu}\cdot\boldsymbol{N}^T[[\boldsymbol{F}\boldsymbol{\nu}]]\right\} \quad \text{along } C \tag{16.25}$$

for the phase interface incoherent in rotations.

The entropy production δ_C^2 remains always non-negative for all thermomechanical processes. This allows us to postulate the kinetic equation, describing motion of the phase interface for all quasistatic processes, in the form

$$V = -\mathcal{F}(\boldsymbol{\nu}\cdot[[\boldsymbol{C}]]\boldsymbol{\nu}), \tag{16.26}$$

where $\mathcal{F}$ is the non-negative definite kinetic function depending on the jump of $\boldsymbol{C}$ at C, i.e. $\mathcal{F}(\varsigma) \geq 0$ for $\varsigma > 0$, where

$$\boldsymbol{C} = \boldsymbol{C}_c \equiv \rho\psi\boldsymbol{A} - \boldsymbol{N}^T\boldsymbol{F} - \boldsymbol{M}^T\boldsymbol{K}$$

for the coherent interface and

$$\boldsymbol{C} = \boldsymbol{C}_i \equiv \rho\psi\boldsymbol{A} - \boldsymbol{N}^T\boldsymbol{F}$$

for the one incoherent in rotations, $\boldsymbol{A} = \mathbf{1} - \boldsymbol{n}\otimes\boldsymbol{n}$, and $\mathbf{1}$ is the 3D unit tensor.

In the nonlinear shell theory the tensors $\boldsymbol{C}_c$ and $\boldsymbol{C}_i$ play the role of *the Eshelby tensors* or *the energy-momentum tensors*. In 3D continuum mechanics Eshelby-type tensors have various applications in the configurational mechanics [4, 23, 31, 36]. In particular, in the linear theory of plates and shells the Eshelby tensor was used to formulate the 2D conservation laws and the path-independent integrals, see [31].

Following [1, 4, 16] we take the kinetic function $\mathcal{F}(\varsigma)$ in the form

$$\mathcal{F}(\varsigma) = \begin{cases} \dfrac{k(\varsigma-\varsigma_0)}{1+\xi(\varsigma-\varsigma_0)} & \varsigma \geq \varsigma_0, \\ 0 & -\varsigma_0 < \varsigma < \varsigma_0, \\ \dfrac{k(\varsigma+\varsigma_0)}{1-\xi(\varsigma+\varsigma_0)} & \varsigma \leq -\varsigma_0. \end{cases} \tag{16.27}$$

Here ς_0 describes the effects associated with nucleation of the new phase and action of the surface tension, see [1], ξ is a parameter describing limit value of the phase transition velocity [4], and k is a positive kinetic factor. Equation (16.26) with (16.27) can be considered as the special constitutive equation describing the motion of phase interfaces in shells.

16.6 Variational Statement of Thermodynamic Equilibrium of Two-Phase Shell

Since the contributions of Gibbs [19], the variational formulation is widely used for description of phase transitions in solids, see e.g. [3, 5, 9, 22, 37]. The weak statement of quasistatic problems of non-linear shells undergoing PT was considered by Eremeyev and Pietraskiewicz [13], where the thermodynamic continuity condition was derived and relations for the Eshelby tensor $\boldsymbol{C}$ in shells were obtained. Let us note that the variational approach requires the thermodynamic equilibrium of two-phase shell to be the minimizer or the stationary point of the functional of total energy. It does not describe the evolution of C as the external loads or the temperature are changing. The evolution of phase interface can be analyzed using the kinetic equation (16.26) and the BVP presented in previous Sections.

Let us consider the isothermal process, i.e. we assume that θ is constant and $\varphi = 0$ during loading. In this case the thermodynamic equilibrium corresponds to the local or global extremum of the functional of free energy [19, 22]. Hence, the free energy ψ plays the role of the strain energy used in [13], and we have the variational principle in the form

$$\delta\mathfrak{U} = \mathfrak{A}, \quad \mathfrak{U} = \iint\limits_{\mathcal{M}} \rho\psi \, da, \tag{16.28}$$

where we use the same relation for $\mathfrak{A}$ as in Sect. 16.3, but here $\boldsymbol{\nu} = \delta\boldsymbol{u}$ is the virtual translation vector and $\boldsymbol{\omega} = \operatorname{ax}(\delta\boldsymbol{Q}\boldsymbol{Q}^T)$ the virtual rotation vector.

Using (16.17) and (16.18), from (16.28) follow the equilibrium equations (16.9) and the static compatibility conditions (16.19). Additionally, we obtain the thermodynamic compatibility condition

$$[[\boldsymbol{\nu} \cdot \boldsymbol{C}\boldsymbol{\nu}]] = 0, \tag{16.29}$$

where $\boldsymbol{C} = \boldsymbol{C}_c$ for the coherent interface and $\boldsymbol{C} = \boldsymbol{C}_i$ for the interface incoherent in rotations. Equation (16.29) can be used to find position of the phase interface C in the thermodynamic equilibrium state. It corresponds to the stationary solution of the kinetic equation (16.26).

Assuming adiabatic behaviour, i.e. when η is constant and $\chi = 0$, we use the variational principle (16.28) with the functional of internal energy [19, 22],

$$\mathfrak{U} = \iint\limits_{\mathcal{M}} \rho\varepsilon \, da.$$

The continuity condition (16.29) was extended in [44] taking into account the phase interface energy and other line fields defined along C.

16.7 Conclusions

We have presented the resultant, two-dimensional thermomechanics of shells undergoing diffusionless, displacive phase transitions of martensitic type of the shell material. We have used the extended thermodynamics of shells with two temperature fields, i.e. the referential surface mean temperature and the temperature deviation, as well as two dual surface entropy related fields. We have discussed the thermodynamic continuity conditions along the curvilinear phase interface for quasistatic motion and for thermodynamic equilibrium.

Summarising, in the case of finite deformations the thermomechanic BVP for thermoelastic shells undergoing phase transition consists of:

- the equilibrium equations $(16.9)_{1,2}$ supplemented by appropriate static and kinematic boundary conditions for $\boldsymbol{N}$, $\boldsymbol{M}$, $\boldsymbol{u}$, and $\boldsymbol{Q}$,
- the thermoconductivity equations (16.13) and (16.14) with appropriate boundary conditions for θ and φ,
- the compatibility conditions (16.19), (16.20), and (16.23) along the interface C,
- the kinetic equation (16.26) or the thermodynamic equilibrium condition (16.29) along C,

all supplemented with the proper constitutive equations for $\boldsymbol{N}$, $\boldsymbol{M}$, ε, η, χ, $\boldsymbol{q}$, and $\boldsymbol{s}$, see [17]. The kinetic equation (16.26) is used to find position of the curvilinear interface C in its quasistatic motion, while (16.29) is used to find the equilibrium position of C.

The BVP summarized above was illustrated in [17] by the 1D analytically solved example of stretching and bending of the two-phase circular plate. The somewhat simpler version of the thermomechanic shell model undergoing PT with only one surface mean temperature field was earlier developed in [15] and illustrated by the 1D analytic solution of tension and bending of two-phase tube in membrane [15] and bending [16] approximations. However, realistic 2D experimental observations on thin-walled samples presented in papers cited in Introduction can only be verified numerically by two-dimensional solutions of the BVP developed here. For this purpose one still needs to develop 2D computer codes based on the extended finite element method (XFEM) for shells with moving singular curves.

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