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On exact expressions of the bending tensor in the nonlinear theory of thin shells

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Abstract: Some equivalent exact expressions of the bending tensor in the nonlinear theory of thin shells are reviewed. It is noted that the bending tensor, proposed by X.Q. Shen, K.T. Li, Y. Ming "The modified model of Koiter's type for the nonlinearly elastic shells", Appl. Math. Mod. 34 (2010) 3527-3535 as a third-degree polynomial of displacements, is an approximate expression, not the exact one. Then integrability of the fourth kinematic boundary condition, associated with two different but equivalent exact expressions of the bending tensor, is briefly discussed. Finally, a few modified definitions of the bending tensor proposed in the literature are reminded. Within the first-approximation theory they all lead to energetically equivalent models of elastic shells.
Keywords: Thin shell, Nonlinear theory, Bending tensor, Boundary rotation

1. Introduction

Discussing a modified nonlinear model of thin elastic shells, Shen et al. [1] proposed exact invariant expressions for the surface strain tensor and the tensor of change of surface curvature, the latter briefly called the bending tensor here. The bending tensor of [1] was then claimed to be "more exact than Ciarlet's expression" defined in the Theorem 10.3-2 of Ciarlet [2].

In this note I first review several equivalent exact expressions of the surface strain measures derived in many earlier papers and books, which were not referred to in [1]. By comparing the results with those proposed in [1] it is seen that the bending tensor, derived in [1] as a third degree polynomial of displacements, is still an approximate expression, not the exact one. Then, I briefly discuss the formulation of the fourth kinematic boundary condition compatible with the two-dimensional principle of virtual work for the shell. It is indicated that the second expression $(4)_2$ is more convenient for the formulation of the fourth kinematic boundary condition. Finally, I remind that the strain energy density of the first-approximation theory of thin elastic shells is itself approximate. Within its error margin several modified definitions of the bending tensor proposed in the literature lead to energetically equivalent nonlinear shell models.

2. Exact expressions of the surface strain measures

Let $\mathbf{r}(\theta^\alpha)$ and $\bar{\mathbf{r}}(\theta^\alpha)$ be the position vectors of the undeformed and deformed base surface M and $\bar{M}$ of the shell, respectively, where $\theta^\alpha, \alpha = 1,2$, are convected curvilinear surface coordinates. At each point of M we have the natural base vectors $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\theta^\alpha \equiv \mathbf{r},_\alpha$, the metric tensor $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ with the determinant $a = \det(a_{\alpha\beta})$, the unit normal vector $\mathbf{n} = (\mathbf{a}_1 \times \mathbf{a}_2)/|\mathbf{a}_1 \times \mathbf{a}_2|$ orienting M, the curvature tensor $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n},_\beta = \mathbf{a}_\alpha,_\beta \cdot \mathbf{n}$, and the permutation tensor $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta)\cdot\mathbf{n}$. The reciprocal base vectors $\mathbf{a}^\alpha$ and the corresponding metric tensor $a^{\alpha\beta}$ are then found from $\mathbf{a}^{\alpha}\cdot\mathbf{a}_{\beta}=\delta^{\alpha}_{\beta}$ and $a^{\alpha\beta}=\mathbf{a}^{\alpha}\cdot\mathbf{a}^{\beta}$, respectively, where δ^{α}_{β} is the Kronecker symbol.

Let $\mathbf{u}=u_{\alpha}\mathbf{a}^{\alpha}+w\mathbf{n}$ be the displacement vector of the surface deformation $M\to\bar{M}$ such that $\bar{\mathbf{r}}=\mathbf{r}+\mathbf{u}$. Then on $\bar{M}$ we can define geometric quantities $\bar{\mathbf{a}}_{\alpha},\bar{a}_{\alpha\beta},\bar{a},\bar{\mathbf{n}},\bar{b}_{\alpha\beta},\bar{\varepsilon}_{\alpha\beta}$ by similar formulas as above. Each barred quantity can then be expressed through the same unbarred quantity and components of $\mathbf{u}$ by explicit formulas presented, for example, in [3-6]. In particular, we have

$$\bar{\mathbf{a}}_{\alpha}=\mathbf{a}_{\alpha}+\mathbf{u},_{\alpha}=l_{\lambda\alpha}\mathbf{a}^{\lambda}+\phi_{\alpha}\mathbf{n}\,,\quad \bar{\mathbf{n}}=\frac{1}{2}\bar{\varepsilon}^{\alpha\beta}\bar{\mathbf{a}}_{\alpha}\times\bar{\mathbf{a}}_{\beta}=n_{\lambda}\mathbf{a}^{\lambda}+n\mathbf{n}\,, \tag{1}$$

where

$$l_{\alpha\beta}=a_{\alpha\beta}+u_{\alpha|\beta}-b_{\alpha\beta}w\,,\quad \phi_{\alpha}=w,_{\alpha}+b^{\lambda}_{\alpha}u_{\lambda}\,,\quad n_{\mu}=\sqrt{\frac{a}{\bar{a}}}\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}\phi_{\alpha}l^{\lambda}_{\cdot\beta}\,,\quad n=\frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}l^{\lambda}_{\cdot\alpha}l^{\mu}_{\cdot\beta}\,, \tag{2}$$

and $(.)_{|\alpha}$ means the covariant surface derivative in the undeformed metric $a_{\alpha\beta}$.

With (1) and (2) the surface strain measures are defined by the following exact expressions in terms of displacements:

$$\gamma_{\alpha\beta}=\frac{1}{2}\left(\bar{a}_{\alpha\beta}-a_{\alpha\beta}\right)=\frac{1}{2}\left(l^{\lambda}_{\cdot\alpha}l_{\lambda\beta}+\phi_{\alpha}\phi_{\beta}-a_{\alpha\beta}\right), \tag{3}$$

$$\begin{aligned}\kappa_{\alpha\beta}&=-\left(\bar{b}_{\alpha\beta}-b_{\alpha\beta}\right)=-n\left(\phi_{\alpha|\beta}+b^{\lambda}_{\beta}l_{\lambda\alpha}\right)-n_{\lambda}\left(l^{\lambda}_{\cdot\alpha|\beta}-b^{\lambda}_{\beta}\phi_{\alpha}\right)+b_{\alpha\beta}\\&=l_{\lambda\alpha}\left(n^{\lambda}{}_{|\beta}-b^{\lambda}_{\beta}n\right)+\phi_{\alpha}\left(n,_{\beta}+b^{\lambda}_{\beta}n_{\lambda}\right)+b_{\alpha\beta}\,.\end{aligned} \tag{4}$$

In (4) the minus sign in front of $(\bar{b}_{\alpha\beta}-b_{\alpha\beta})$ is just conventional here and may differ in different papers. The first formula of (4) has been calculated using the definition $\bar{b}_{\alpha\beta}=-\bar{\mathbf{a}},_{\alpha}\cdot\bar{\mathbf{n}},_{\beta}$, while the second one of (4) applying the equivalent definition $\bar{b}_{\alpha\beta}=\bar{\mathbf{a}}_{\alpha},_{\beta}\cdot\bar{\mathbf{n}}$.

As indicated in my survey article [6], the exact invariant formula (3) and the first one of (4) for the surface strain measures were originally proposed in different but equivalent forms by Mushtari [7] and then used in many Russian papers partly summarized by Galimov [8-10]. In the English literature different exact expressions equivalent to (3) and $(4)_1$ were proposed by Leonard [11], Sanders [12], and Koiter [3], which were then used in a number of later publications, for example [13-15,4,5].

The quadratic polynomial of displacements given by (3) for the strain tensor $\gamma_{\alpha\beta}$ is equivalent to that proposed in (3.1) of [1]. The exact formula $(4)_1$ for the bending tensor is expressed through the fields n_{μ},n which, according to (2), contain the square-root invariant $\sqrt{a/\bar{a}}$, where

$$\frac{\bar{a}}{a}=\frac{1}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\kappa}\bar{a}_{\alpha\beta}\bar{a}_{\lambda\kappa}=1+2\gamma^{\alpha}_{\alpha}+2\left(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta}-\gamma^{\beta}_{\alpha}\gamma^{\alpha}_{\beta}\right). \tag{5}$$

Thus, with (3) it follows that $\bar{a}/a$ is the forth-degree polynomial of displacements which cannot, in general, be exactly represented as a quadratic polynomial taken to the second power. As a result, $\sqrt{a/\bar{a}}$ is a non-rational function of displacements, in general, and so is the formula $(4)_1$ for $\kappa_{\alpha\beta}$. The bending tensor $R_{\alpha\beta}$, derived in (3.28) of [1] as the third degree polynomial of displacements, cannot be equivalent to $(4)_1$ and must be approximate, not exact one. It seems that the error in [1] was made already in the Lemma 1, where $\bar{a}/a$ was found to be the second degree polynomial of displacements, which is obviously incorrect.

3. Formulation of the fourth kinematic boundary condition

In the nonlinear theory of thin shells the surface strain measures are usually introduced into the two-dimensional principle of virtual work formulated on M to generate three equilibrium equations as well as four work-conjugate natural static and kinematic boundary conditions. When the exact formulas (3) and $(4)_1$ are used for this purpose, from detailed transformations performed for example by Galimov [16] and Pietraszkiewicz [13,5] it follows that along the shell boundary contour ∂M some boundary couple should perform the virtual work on a virtual rotation about tangent to the deformed shell boundary contour. This virtual rotation was found to be $\bar{\mathbf{v}} \cdot \delta\bar{\mathbf{n}}$ in [16], $(\bar{\mathbf{n}} \cdot \delta\mathbf{u}),_{\nu}$ in [13], and $\bar{\mathbf{r}}' \cdot \delta\mathbf{\Omega}_t$ in [5], where $\bar{\mathbf{v}}$ and $\mathbf{v}$ are the outward unit normal vectors to the deformed and undeformed boundaries $\partial\bar{M}$ and ∂M, respectively, $(.),_{\nu} = (.),_{\alpha}\nu^{\alpha}$, $\nu^{\alpha} = \mathbf{v} \cdot \mathbf{a}^{\alpha}$, $\bar{\mathbf{r}}' = d\bar{\mathbf{r}}/ds$, s is the length parameter along ∂M, and $\delta\mathbf{\Omega}_t$ is the virtual rotation vector of the shell boundary contour.

At that time it was not apparent what type of a scalar function should be prescribed along ∂M in order to satisfy the virtual rotational boundary constraint $\bar{\mathbf{v}} \cdot \delta\bar{\mathbf{n}} = 0$, or $(\bar{\mathbf{n}} \cdot \delta\mathbf{u}),_{\nu} = 0$, or $\bar{\mathbf{r}}' \cdot \delta\mathbf{\Omega}_t = 0$. Only some years later we treated in [17] the virtual rotation expressions discussed above as differential one-forms on a suitably defined six-dimensional manifold of displacement derivatives $\mathbf{u}', \mathbf{u},_{\nu}$. It was found in [17] that all these expressions and some other ones available in the literature are not integrable. This means that neither of them, even multiplied by an integrating factor $\mu(\mathbf{u}', \mathbf{u},_{\nu})$, can be represented in the form $\delta\varphi(\mathbf{u}', \mathbf{u},_{\nu})$. This property of all such virtual rotations does not allow to directly formulate the fourth kinematic boundary condition for the so constructed nonlinear shell models. Additional nontrivial transformations along the shell boundary ∂M suggested in [17] had to be performed in order to overcome this difficulty and to formulate the correct fourth kinematic boundary condition of the nonlinear shell BVP.

In order to avoid the above problem following directly from the first exact expression $(4)_1$, Pietraszkiewicz and Szwabowicz [18] proposed to apply the alternative exact formula $(4)_2$. When $(4)_2$ is introduced into the principle of virtual work on M, it generates the virtual rotation $\delta(\bar{\mathbf{n}} \cdot \mathbf{v})$ along ∂M, see [19], eq. (2.26). If treated as the differential one-form of $\mathbf{u}', \mathbf{u},_{\nu}$ along ∂M this virtual rotation is obviously integrable, $\delta(\bar{\mathbf{n}} \cdot \mathbf{v}) \equiv \delta n_{\nu}$, where $n_{\nu} = \bar{\mathbf{n}} \cdot \mathbf{v}$. Hence, the virtual boundary constraint $\delta n_{\nu} = 0$ allows one to formulate the fourth kinematic boundary condition in the form $n_{\nu} = n_{\nu}^{*}$, where n_{ν}^{*} is an assumed value of n_{ν} along ∂M. The exact formula $(4)_2$ for $\kappa_{\alpha\beta}$ was then used in a number of papers, for example [19-26].

The above discussion indicates that, although both exact expressions (4) of the bending tensor are algebraically equivalent, the second one $(4)_2$ is more convenient in deriving directly the complete set of work-conjugate static and kinematic boundary conditions of the nonlinear theory of thin shells. Analyzing the expressions (3.22) and (3.23) of [1] it is apparent that the bending tensor of [1] is some approximate form of our first expression $(4)_1$, not of the second one $(4)_2$. Thus, the approximate bending tensor of [1] will not allow the authors to directly formulate the fourth kinematic boundary condition compatible with the principle of virtual work of the nonlinear thin shell theory.

The problem of integrability of the virtual rotational constraint along ∂M is avoided when the shell boundary is simply supported (where the boundary couple is zero) or entirely clamped (where the virtual rotation is zero). Exactly such boundary conditions are assumed in almost all theoretical and numerical analyses of the nonlinear thin shell structures, also in the numerical example discussed in [1]. Among a few exceptions I mention here two papers by Opoka and Pietraszkiewicz [27,28], where the kinematic boundary conditions were carefully discussed. Another example of a careful construction of the kinematic boundary conditions following from the bending tensor $K_{\alpha\beta}$ of [29] is provided by Libai and Simmonds [30].

4. Energetic equivalence of the first-approximation shell models

When small strains are assumed in the shell space, the constitutive equations follow by differentiating the strain energy density $\Sigma = \Sigma(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$. To within the first approximation the density becomes the sum of two quadratic functions describing the stretching and bending energies of the shell base surface. The accuracy of such an approximation was discussed in a number of papers reviewed in section 3.4 of [6]. According to Koiter [2], this density can be presented in the form

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu} \left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12} \kappa_{\alpha\beta}\kappa_{\lambda\mu} \right) + O\left(Eh\eta^2\theta^2 \right), \tag{6}$$

where h is the undeformed shell thickness, $H^{\alpha\beta\lambda\mu}$ are components of the modified elasticity tensor, E is the Young modulus, η is the largest strain in the shell space, and θ is the small parameter defined in [31] as the maximal value of five different small parameters appearing in thin shell theory.

Within the error of (6), alternative definitions of the surface bending tensor, which differ from (4) by small terms such as $b_{\alpha\beta}\gamma_\lambda^\lambda$ or $b_\alpha^\lambda \gamma_{\lambda\beta}$, may be regarded to be energetically equivalent to that given in (4), for example

$$\rho_{\alpha\beta} = \left(\bar{b}_{\alpha\beta} - b_{\alpha\beta} \right) - \frac{1}{2}\left(b_\alpha^\lambda \gamma_{\lambda\beta} + b_\beta^\lambda \gamma_{\alpha\lambda} \right), \quad \chi_{\alpha\beta} = -\left(\sqrt{\frac{\bar{a}}{a}}\, \bar{b}_{\alpha\beta} - b_{\alpha\beta} \right) + b_{\alpha\beta}\gamma_\lambda^\lambda\,,$$

$$\rho^*_{\alpha\beta} = \sqrt{\frac{\bar{a}}{a}}\, \bar{b}_{\alpha\beta} - b_{\alpha\beta}\,, \quad R^\#_{\alpha\beta} = \frac{1}{\sqrt{a}} \bar{b}_{\alpha\beta} - b_{\alpha\beta}\,, \quad K_{\alpha\beta} = -\left(\sqrt{\frac{\bar{a}}{a}}\, \bar{b}_{\alpha\beta} - b_{\alpha\beta} \right) + b_{\alpha\beta}\gamma_\lambda^\lambda + \frac{1}{2}\left(b_\alpha^\lambda \gamma_{\lambda\beta} + b_\beta^\lambda \gamma_{\alpha\lambda} \right). \tag{7}$$

With the displacemental expression $(4)_1$ the tensor $\rho_{\alpha\beta}$ was used in [3,13], $\rho^*_{\alpha\beta}$ was proposed in [3], while $K_{\alpha\beta}$ was proposed and used in [29]. With the displacemental expression $(4)_2$ the tensor $\chi_{\alpha\beta}$ was proposed in [18] and used in [21,32,25], while $\rho_{\alpha\beta}$ was applied in [33]. Within the error of (6) the tensor $R^\#_{\alpha\beta}$ proposed by Ciarlet [2] may also be regarded as energetically equivalent to $(4)_1$. Each of the energetically equivalent definitions of the bending tensor has some distinctive features. For example, $\rho_{\alpha\beta}$ and $K_{\alpha\beta}$, when linearised, reduce to the "best" bending measure of the linear shell theory according to [34], $\chi_{\alpha\beta}$, $K_{\alpha\beta}$, and $\rho^*_{\alpha\beta}$ are the third degree polynomials of displacements, while $R^\#_{\alpha\beta}$ are well defined for all smooth fields $\mathbf{u}(\theta^\alpha)$ irrespective of whether or not the vectors $\boldsymbol{a}_\alpha$ are linearly dependent.

In light of the above arguments, the statement by Shen et al. [1] that the modified shell model based on their bending tensor $R_{\alpha\beta}$ is better than Ciarlet's model is not justified.

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