

# Material symmetry group of the non-linear polar-elastic continuum

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## Abstract

We extend the material symmetry group of the non-linear polar-elastic continuum by taking into account microstructure curvature tensors as well as different transformation properties of polar and axial tensors. The group consists of an ordered triple of tensors which makes the strain energy density of polar-elastic continuum invariant under change of reference placement. An analogue of the Noll rule is established. Four simple specific cases of the group with corresponding reduced forms of the strain energy density are discussed. Definitions of polar-elastic fluids, solids, liquid crystals and subfluids are given in terms of members of the symmetry group. Within polar-elastic solids we discuss in more detail isotropic, hemitropic, cubic-symmetric, transversely isotropic, and orthotropic materials and give explicitly corresponding reduced representations of the strain energy density. For physically linear polar-elastic solids, when the density becomes a quadratic function of strain

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measures, reduced representations of the density are established for monoclinic, orthotropic, cubic-symmetric, hemitropic and isotropic materials in terms of appropriate joint scalar invariants of stretch, wryness and undeformed structure curvature tensors.

*Keywords:* Material symmetry, Symmetry group, Polar-elastic continuum, Cosserat continuum, Non-linear theory, Constitutive equations

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## 1. Introduction

Material symmetry group of the non-linear micropolar continuum was first characterized by Eringen and Kafadar (1976). They discussed all density-preserving deformations and all microrotations of reference placement of the continuum that cannot be experimentally detected. Simple micropolar solids and simple micropolar fluids in the spatial description were defined in terms of members of the group. After 35 years we are still not aware of any detailed discussion of the material symmetry group of non-linear micropolar continuum and of appropriate reduction of constitutive equations in terms of members of the group.

In the present report we restrict ourselves to the non-linear polar-elastic continuum, which material behaviour is described by the strain energy density  $W$  per unit reference volume. As compared with Eringen and Kafadar (1976) we introduce three modifications:

1. At each material point our density  $W$ , satisfying the principle of material frame-indifference, depends explicitly not only on natural Lagrangian stretch  $\mathbf{E}$  and wryness  $\mathbf{\Gamma}$  tensors, but additionally upon the reference microstructure curvature tensor  $\mathbf{B}$  as a parametric tensor. Eringen and Kafadar

(1976) used similar strain measures in  $W$ , but introduced referential mass density  $\rho_{\varkappa}$  and microinertia tensor  $\mathbf{J}_{\varkappa}$  as parametric quantities.

2. In discussing invariance properties of  $W$  we take into account that, while  $\mathbf{E}$  is a polar tensor,  $\mathbf{\Gamma}$  and  $\mathbf{B}$  are axial tensors which change signs under inversion transformation (mirror reflection) of 3D space. Eringen and Kafadar (1976) did not take into account that their  $\mathbf{\Gamma}$  was the axial tensor.

3. Our material symmetry group  $\mathcal{G}_{\varkappa}$  consists of ordered triple of tensors: unimodular  $\mathbf{P}$ , orthogonal  $\mathbf{R}$ , and second-order  $\mathbf{L}$ . These tensors appear from transformation of  $\mathbf{E}$ ,  $\mathbf{\Gamma}$  and  $\mathbf{B}$  under an arbitrary change of reference placement of the micropolar body. The transformation properties of  $\mathbf{B}$  are quite different from those of  $\mathbf{J}_{\varkappa}$ .

As a result of these modifications, our material symmetry group  $\mathcal{G}_{\varkappa}$  does not coincide with the group introduced by Eringen and Kafadar (1976).

The tensor  $\mathbf{B}$  appears naturally during description of the reference placement, see Pietraszkiewicz and Eremeyev (2009). The case  $\mathbf{B} \neq \mathbf{0}$  indicates non-uniform distribution of directors in the reference placement. From the mathematical point of view the case  $\mathbf{B} \neq \mathbf{0}$  relates to non-Euclidean geometry of polar material because the directors can be considered as a nonholonomic basis. From the physical point of view necessity of taking into account  $\mathbf{B}$  in constitutive equations corresponds to proper description of microstructure of materials.  $\mathbf{B}$  can be considered as an analogue of the curvature tensor in the theory of shells or of material parameters describing helical substructures of rods, see for example Eremeyev and Pietraszkiewicz (2006); Lauderdale and O'Reilly (2007).

Within Cosserat-type theories of shells and rods various definitions of ma-

terial symmetry groups were proposed for example by Murdoch and Cohen (1979); Eremeyev and Pietraszkiewicz (2006); Luo and O'Reilly (2000); Luderale and O'Reilly (2007). The material symmetry group of the second-grade elastic materials was considered for example by Huang and Smith (1967); Elżanowski and Epstein (1992). Huang and Smith (1967) took into account difference between axial and polar tensors in construction of the material symmetry group. Let us note that the invariance of the strain energy density under transformations of the reference placement is similar but not the same as the uniformity and homogeneity properties considered by Epstein and de Leon (1996, 1998).

The paper is organized as follows. In Section 2 we recall after Pietraszkiewicz and Eremeyev (2009) some basic relations of non-linear micropolar continuum. In particular, we remind definitions of stretch  $\mathbf{E}$  and wryness  $\mathbf{\Gamma}$  tensors of undeformed structure curvature tensor  $\mathbf{B}$ , of strain energy density  $W$  satisfying the principle of material fame-indifference, of referential  $\mathbf{S}$ ,  $\mathbf{K}$  and Eulerian  $\mathbf{T}$ ,  $\mathbf{M}$  stress and couple-stress tensors, as well as appropriate forms of equilibrium conditions.

Transformation properties of various fields under change of reference placement are discussed in Section 3. Then in Section 4 invariance requirements for  $W$  under change of reference placement are analysed. This allows one to derive the material symmetry group  $\mathcal{G}_\varkappa$  of the polar-elastic continuum. The group  $\mathcal{G}_\varkappa$  consists of an ordered triple of tensors which make  $W$  invariant under change of reference placement. For  $\mathcal{G}_\varkappa$  we establish an analogue of Noll's rule, see Noll (1958).

Four specific cases of  $\mathcal{G}_\varkappa$  with corresponding considerably simplified forms

of  $W$  are discussed in Section 5. After introducing in Section 6 property of isotropy of the polar-elastic material, polar-elastic fluids are defined by requiring  $\mathcal{G}_x$  to be the maximal group. It is found that polar-elastic fluids are always isotropic and their strain energy density  $W$  is the hemitropic scalar-valued even function of the structure curvature tensor  $\mathbf{C}$  of deformed placement. By the representation theorem of Smith and Smith (1971) the density  $W$  is reduced to the scalar-valued function of the deformed density  $\rho$  and of six scalar invariants of  $\mathbf{C}$ . The constitutive equations for  $\mathbf{T}$  and  $\mathbf{M}$  are derived, which generalize the ones proposed earlier for polar-elastic fluids for example by Aero et al. (1965); Eringen (1966, 1997); Allen et al. (1967).

In Section 8 we define the group  $\mathcal{G}_x$  of polar-elastic solids using an additional hypothesis that elements of  $\mathcal{G}_x$  are described by orthogonal tensors which are the same for the position vector and the directors. Polar-elastic liquid crystals and subfluids are then defined in Section 9 by  $\mathcal{G}_x$  which contains some elements not belonging to the orthogonal group. Examples of Kelvin's medium defined by Grekova and Zhilin (2001) and of Ericksen's liquid crystal proposed by Eringen (1997, 2001) are briefly discussed.

Simplified representations of  $W$  for some forms of anisotropy of polar-elastic solids are discussed in Section 10.

Applying representation theorems of the scalar-valued tensor functions derived by Spencer (1965, 1971), we analyse possible reduction of  $W$  for isotropic, hemitropic, cubic-symmetric, orthotropic, and transversely isotropic non-linear polar-elastic solids. As in Section 6 we apply the theory of representation of scalar-valued functions of several tensorial arguments summarised in Spencer (1971); Bohler (1987); Smith (1994); Zheng (1994) where

other references can be found. For polar-elastic solids the material symmetry group is described by orthogonal transformations alone. This gives possibility to represent  $W$  in terms of joint invariants of the strain measures  $\mathbf{E}$  and  $\mathbf{\Gamma}$  as well as of the structure curvature tensor  $\mathbf{B}$ .

Finally, in Sections 11 and 12 physically linear polar-elastic solids are analysed under mirror reflection as well as under monoclinic, orthotropic, cubic, hemitropic, and isotropic symmetry conditions. For each case of symmetries reduced forms of  $W$  in terms of appropriate joint scalar invariants of tensors  $\mathbf{E}$ ,  $\mathbf{\Gamma}$ , and  $\mathbf{B}$  are given. In Appendix we present table with invariants for the isotropic symmetry.

## 2. Some relations of the non-linear micropolar elastic continuum

Let the body  $\mathcal{B}$  consisting of material particles  $X, Y, \dots$  deform in three-dimensional (3D) Euclidean physical space  $\mathcal{E}$  which translation vector space is  $E$ . According to Cosserat and Cosserat (1909); Truesdell and Toupin (1960); Eringen and Kafadar (1976) for example, each material particle  $X \in \mathcal{B}$  of the polar-elastic continuum has six degrees of freedom of rigid body.

In the reference (undeformed) placement  $\varkappa(\mathcal{B}) = B_\varkappa \subset \mathcal{E}$  position  $x \in \mathcal{E}$  of  $X \in \mathcal{B}$  is given by the vector  $\mathbf{x} \in E$  relative to origin  $o \in \mathcal{E}$  of an inertial frame  $(o, \mathbf{i}_a)$ , where  $\mathbf{i}_a \in E$ ,  $a = 1, 2, 3$ , is a right-handed triple of orthonormal vectors. Orientation of  $X \in \mathcal{B}$  in  $E$  is fixed by the right-handed triple of orthonormal directors  $\mathbf{h}_a \in E$ , so that  $\mathbf{h}_a = \mathbf{H}\mathbf{i}_a$ , where  $\mathbf{H} = \mathbf{h}_a \otimes \mathbf{i}_a \in Orth^+$  is the proper orthogonal structure tensor of  $B_\varkappa$ ,  $\mathbf{H}^{-1} = \mathbf{H}^T$ ,  $\det \mathbf{H} = +1$ , and  $\otimes$  denotes the tensor product.

In the actual (deformed) placement  $\gamma(\mathcal{B}) = B_\gamma = \chi(B_\varkappa) \in \mathcal{E}$ ,  $\chi =$

$\gamma \circ \varkappa^{-1}$ , position  $y \in B_\varkappa$  of  $X \in \mathcal{B}$  becomes defined by the vector  $\mathbf{y} \in E$  taken here relative to the same origin  $o \in \mathcal{E}$ . Orientation of  $X$  becomes fixed by the right-handed triple of orthonormal directors  $\mathbf{d}_a \in E$  or by the structure tensor  $\mathbf{D} = \mathbf{d}_a \otimes \mathbf{i}_a \in Orth^+$  of  $B_\gamma$ . As a result, the finite displacement of polar-elastic continuum can be described by two smooth mappings (see Fig. 1):

$$\mathbf{y} = \chi(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}), \quad \mathbf{d}_a = \mathbf{Q}(\mathbf{x})\mathbf{h}_a, \quad (1)$$

where  $\mathbf{u} \in E$  is the translation vector and  $\mathbf{Q} = \mathbf{D}\mathbf{H}^T = \mathbf{d}_a \otimes \mathbf{h}_a \in Orth^+$  is the proper orthogonal microrotation tensor,  $\mathbf{Q}^{-1} = \mathbf{Q}^T$ ,  $\det \mathbf{Q} = +1$ . Two independent fields  $\mathbf{u}(\mathbf{x})$  and  $\mathbf{Q}(\mathbf{x})$  describe translational and rotational degrees of freedom of polar-elastic continuum.

Natural Lagrangian relative stretch and wryness (or change of microstructure orientation) tensors  $\mathbf{E}$  and  $\mathbf{\Gamma}$  are defined according to Pietraszkiewicz and Eremeyev (2009) as

$$\mathbf{E} = \mathbf{Q}^T \mathbf{F} - \mathbf{I}, \quad \mathbf{\Gamma} = -\frac{1}{2} \mathbf{E} : (\mathbf{Q}^T \text{Grad } \mathbf{Q}). \quad (2)$$

Here  $\mathbf{F} = \text{Grad } \mathbf{y}$ ,  $\det \mathbf{F} > 0$ , is the classical deformation gradient tensor taken relative to  $B_\varkappa$ ,  $\mathbf{I}$  is the identity (metric) tensor of 3D space  $E$ ,  $\mathbf{E} = -\mathbf{I} \times \mathbf{I}$  is the 3rd-order skew permutation tensor with  $\times$  the vector product, while the double dot product  $:$  of two 3rd-order tensors  $\mathbf{A}$ ,  $\mathbf{B}$  represented in  $\mathbf{h}_a$  is defined as  $\mathbf{A} : \mathbf{B} = A_{amn} B_{mnb} \mathbf{h}_a \otimes \mathbf{h}_b$ . The strain measures defined in (2) are not symmetric, in general,  $\mathbf{E}^T \neq \mathbf{E}$ ,  $\mathbf{\Gamma}^T \neq \mathbf{\Gamma}$ .

Gradient of vector field  $\mathbf{v}(\mathbf{x}) \in E$  is a 2nd-order tensor field  $\text{Grad } \mathbf{v} \in E \otimes E$  and gradient of 2nd-order tensor field  $\mathbf{A} \in E \otimes E$  is a 3rd-order tensor

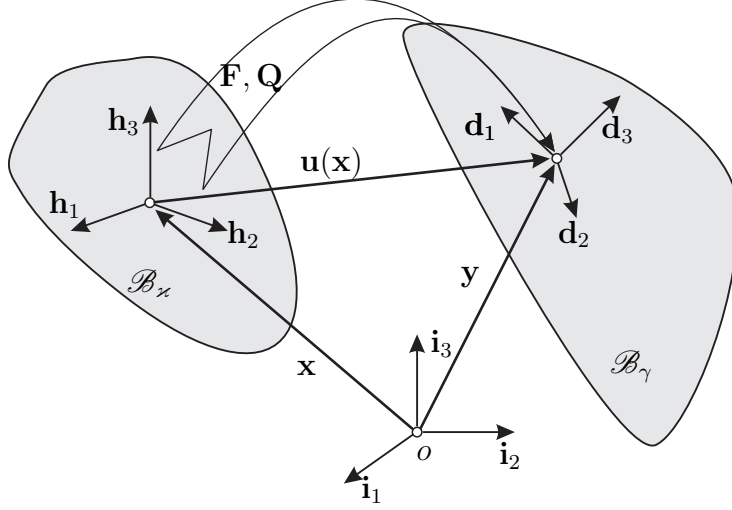


Figure 1: Micropolar body deformation.

field  $\text{Grad } \mathbf{A} \in E \otimes E \otimes E$  defined by (see for example Ogden (1984))

$$\begin{aligned} [\text{Grad} \mathbf{v}(\mathbf{x})] \mathbf{a} &= \frac{d}{dt} \mathbf{v}(\mathbf{x} + t\mathbf{a})|_{t=0}, \\ [\text{Grad} \mathbf{A}(\mathbf{x})] \mathbf{a} &= \frac{d}{dt} \mathbf{A}(\mathbf{x} + t\mathbf{a})|_{t=0}, \quad \text{for any } t \in R, \mathbf{a} \in E. \end{aligned} \quad (3)$$

In particular, gradient of product of 2<sup>nd</sup>-order tensor  $\mathbf{A}(\mathbf{x})$  and vector  $\mathbf{v}(\mathbf{x})$  fields is calculated according to

$$\text{Grad} (\mathbf{A}\mathbf{v}) = \mathbf{v}\text{Grad} \mathbf{A}^T + \mathbf{A}\text{Grad} \mathbf{v}. \quad (4)$$

The wryness tensor  $\mathbf{\Gamma}$  can also be expressed in several equivalent forms, see Pietraszkiewicz and Eremeyev (2009), for example

$$\mathbf{\Gamma} = -\frac{1}{2} \mathbf{h}_a \times (\mathbf{h}_a \mathbf{Q}^T \text{Grad} \mathbf{Q}) = \mathbf{Q}^T \mathbf{C} \mathbf{F} - \mathbf{B}, \quad (5)$$

where  $\mathbf{B}$  and  $\mathbf{C}$  are the respective microstructure curvature tensors of polar continuum in the reference and actual placements defined by

$$\mathbf{B} = \frac{1}{2} \mathbf{h}_a \times \text{Grad} \mathbf{h}_a, \quad \mathbf{C} = \frac{1}{2} \mathbf{d}_a \times \text{grad} \mathbf{d}_a, \quad (6)$$



with the operator grad being taken in the deformed placement  $B_\gamma$ .

In what follows  $\mathbf{B}$  and  $\mathbf{C}$  play an important role because they characterize non-uniform distributions of directors  $\mathbf{h}_a$  and  $\mathbf{d}_a$  in the reference and actual placements, respectively. In particular, if  $\mathbf{h}_a$  are constant in space then  $\mathbf{B} = \mathbf{0}$ .

Material behaviour of the micropolar (hyper)elastic continuum is described by the strain energy density  $W_\varkappa$  per unit volume of undeformed placement  $B_\varkappa$ . The density  $W_\varkappa$  satisfying the principle of material frame-indifference takes the reduced form

$$W_\varkappa = \widehat{W}_\varkappa(\mathbf{E}, \mathbf{\Gamma}; \mathbf{x}, \mathbf{B}). \quad (7)$$

We call the polar-elastic continuum *homogeneous* if there exists a reference placement  $B_\varkappa$  such that  $W_\varkappa$  does not depend on  $\mathbf{x}$  and *materially uniform* if  $W_\varkappa$  does not depend on  $\mathbf{B}$  or  $\mathbf{B} \equiv \mathbf{0}$ .

Constitutive equations for referential stress  $\mathbf{S}$  and couple-stress  $\mathbf{K}$  tensors introduced in Pietraszkiewicz and Eremeyev (2009) are now defined as

$$\mathbf{S} = \frac{\partial W_\varkappa}{\partial \mathbf{E}}, \quad \mathbf{K} = \frac{\partial W_\varkappa}{\partial \mathbf{\Gamma}}. \quad (8)$$

It is apparent that  $\mathbf{S}$  and  $\mathbf{K}$  are not symmetric, in general,  $\mathbf{S}^T \neq \mathbf{S}$ ,  $\mathbf{K}^T \neq \mathbf{K}$ .

The local Lagrangian equilibrium equations as well as kinematic and dynamic boundary conditions of the non-linear theory of micropolar continuum are, see Pietraszkiewicz and Eremeyev (2009), Appendix,

$$\text{Div } \mathbf{T}_N + \rho_\varkappa \mathbf{f} = \mathbf{0}, \quad (9)$$

$$\text{Div } \mathbf{M}_N + \text{ax}(\mathbf{T}_N^T \mathbf{F}^T - \mathbf{F} \mathbf{T}_N) + \rho_\varkappa \mathbf{c} = \mathbf{0} \quad \text{in } B_\varkappa, \quad (10)$$

$$\mathbf{x} - \mathbf{x}^0 = \mathbf{0}, \quad \mathbf{Q} - \mathbf{Q}^0 = \mathbf{0} \quad \text{along } \partial B_{\varkappa d}. \quad (11)$$

$$\mathbf{n}\mathbf{T}_N - \mathbf{t}^0 = \mathbf{0}, \quad \mathbf{n}\mathbf{M}_N - \mathbf{m}^0 = \mathbf{0} \quad \text{along } \partial B_{\varkappa f}. \quad (12)$$

In Eqs. (9) and (10), divergence operator  $\text{Div}$  in  $B_{\varkappa}$  is defined as, for example, in Billington (1986) by

$$[\text{Div } \mathbf{A}(\mathbf{x})]\mathbf{a} = \text{Div} [\mathbf{A}(\mathbf{x})\mathbf{a}], \quad \forall \mathbf{A} \in E \otimes E, \quad \forall \mathbf{a} \in E. \quad (13)$$

In Eqs.(9)–(12),  $\mathbf{T}_N, \mathbf{M}_N \in E \otimes E$  are nominal stress and couple-stress tensors following from the generalised Cauchy theorem  $\mathbf{t}_{(n)} = \mathbf{n}\mathbf{T}_N$ ,  $\mathbf{m}_{(n)} = \mathbf{n}\mathbf{M}_N$ , in which  $\mathbf{t}_{(n)}$  and  $\mathbf{m}_{(n)}$  are surface traction and moment vectors applied at any point of boundary  $\partial P_{\gamma}$  of  $P_{\gamma} \subset B_{\gamma}$ , but measured per unit area of  $\partial P_{\varkappa}$ , respectively, with  $\mathbf{n}$  being the unit vector externally normal to  $\partial P_{\varkappa}$ . The nominal tensors  $\mathbf{T}_N, \mathbf{M}_N$  are related to the referential tensors  $\mathbf{S}$  and  $\mathbf{K}$  given in (8) by

$$\mathbf{T}_N = \mathbf{S}^T \mathbf{Q}^T, \quad \mathbf{M}_N = \mathbf{K}^T \mathbf{Q}^T. \quad (14)$$

Additionally in (9)–(12),  $\mathbf{f}, \mathbf{c} \in E$  are the external force and couple vectors applied at any point of  $B_{\gamma}$  but measured per unit mass of  $B_{\varkappa}$ ,  $\rho_{\varkappa}$  is the mass density in  $B_{\varkappa}$ , while  $\mathbf{t}^0, \mathbf{m}^0 \in E$  are the external boundary force and couple vectors applied on part of the deformed boundary  $\partial B_{\gamma f}$  but measured per unit area of  $\partial B_{\varkappa f}$ , respectively. The vector  $\mathbf{x}^0$  and the tensor  $\mathbf{Q}^0$  given on  $\partial B_{\varkappa d}$  describe translation and rotation of particles prescribed on  $\partial B_{\varkappa d}$ , respectively, where  $\partial B_{\varkappa} = \partial B_{\varkappa d} \cup \partial B_{\varkappa f}$ ,  $\partial B_{\varkappa d} \cap \partial B_{\varkappa f} = \emptyset$ .

Formally different than (9)–(12) but equivalent coordinate-free forms of local equilibrium conditions follow by applying alternative definitions of gradient and divergence operators as well as of the Cauchy theorem, see for

example Maugin (1974, 1998); Scarpetta (1989); Lurie (1990); Zubov (1997); Yeremeyev and Zubov (1999); Dai (2003).

The local Eulerian equilibrium equations and dynamic boundary conditions in the actual placement  $B_\gamma$  corresponding to (9)–(12) are

$$\operatorname{div} \mathbf{T} + \rho \mathbf{f} = \mathbf{0}, \quad \operatorname{div} \mathbf{M} + \operatorname{ax}(\mathbf{T}^T - \mathbf{T}) + \rho \mathbf{c} = \mathbf{0} \quad \text{in } B_\gamma, \quad (15)$$

$$\bar{\mathbf{n}} \mathbf{T} - \bar{\mathbf{t}}^0 = \mathbf{0}, \quad \bar{\mathbf{n}} \mathbf{M} - \bar{\mathbf{m}}^0 = \mathbf{0} \quad \text{along } \partial B_{\gamma f}. \quad (16)$$

In (15) and (16),  $\mathbf{T}$  and  $\mathbf{M}$  are Cauchy-type stress and couple-stress tensors which are related to the referential stress measures by

$$\mathbf{T} = (\det \mathbf{F})^{-1} \mathbf{F} \mathbf{T}_N = (\det \mathbf{F})^{-1} \mathbf{F} \mathbf{S}^T \mathbf{Q}^T, \quad (17)$$

$$\mathbf{M} = (\det \mathbf{F})^{-1} \mathbf{F} \mathbf{M}_N = (\det \mathbf{F})^{-1} \mathbf{F} \mathbf{K}^T \mathbf{Q}^T, \quad (18)$$

$\rho$  is the mass density in  $B_\gamma$ ,  $\operatorname{div}$  is the divergence operator in  $B_\gamma$  defined as in (13),  $\bar{\mathbf{t}}^0$ ,  $\bar{\mathbf{m}}^0$  are measured per unit area of  $\partial B_{\gamma f}$ , with  $\bar{\mathbf{n}}$  being the unit vector externally normal to  $\partial B_\gamma$ .

### 3. Change of reference placement

Let us introduce another reference placement  $\varkappa_*(\mathcal{B}) = B_* \in \mathcal{E}$  of  $\mathcal{B}$ , in which position  $x_* \in B_*$  of  $X \in \mathcal{B}$  is given by the vector  $\mathbf{x}_*$  relative to the same origin  $o \in \mathcal{E}$  and its orientation is fixed by three right-handed orthonormal directors  $\mathbf{h}_{*a}$  (Fig. 2). Let  $\mathbf{P}$ ,  $\det \mathbf{P} > 0$ , be the deformation gradient transforming  $d\mathbf{x}$  into  $d\mathbf{x}_*$ , and  $\mathbf{R} \in \operatorname{Orth}^+$  be the rotation tensor transforming  $\mathbf{h}_a$  into  $\mathbf{h}_{*a}$ , so that

$$d\mathbf{x}_* = \mathbf{P} d\mathbf{x}, \quad \mathbf{h}_{*a} = \mathbf{R} \mathbf{h}_a. \quad (19)$$

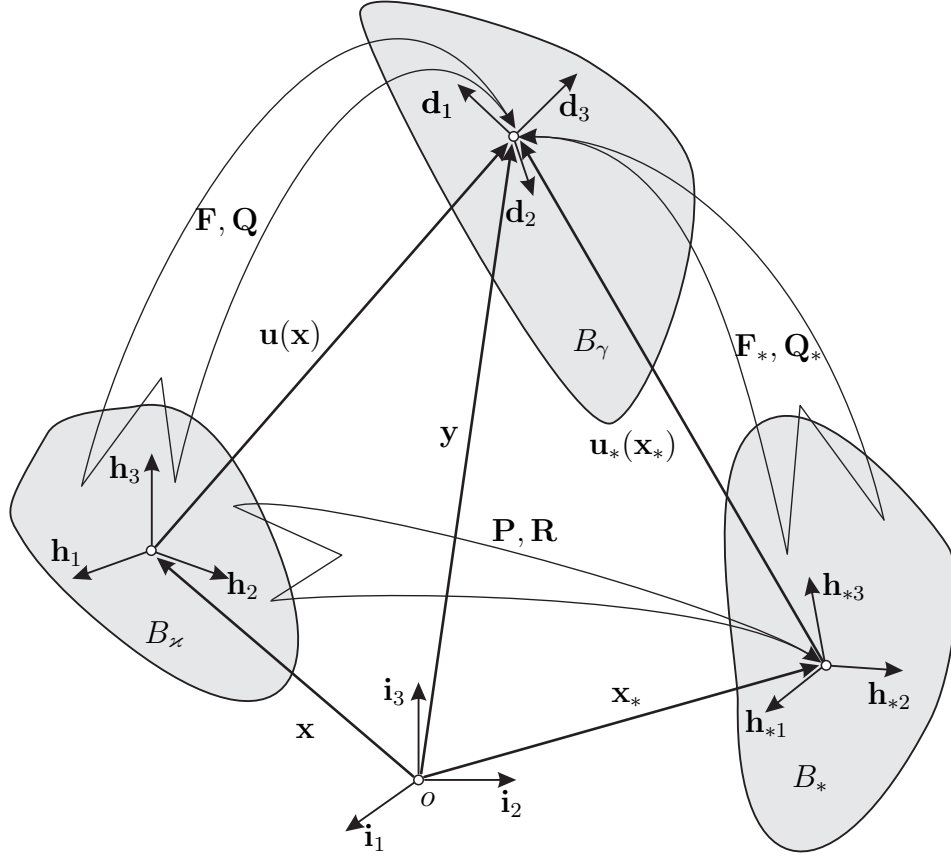


Figure 2: Change of reference placement.

In what follows all fields associated with deformation relative to  $B_*$  will be marked by lower index  $*$ .

Let us analyse how transform the strain measures  $\mathbf{E}$  and  $\mathbf{\Gamma}$  under change of reference placement  $B_x \rightarrow B_*$ .

Since  $dy = \mathbf{F}dx = \mathbf{F}_*dx_*$ , where  $\mathbf{F}_* = \text{Grad}_* \mathbf{y}(\mathbf{x}_*)$ ,  $\det \mathbf{F}_* > 0$ , is the deformation gradient relative to  $B_*$  and rotation  $\mathbf{h}_{*a}$  into  $\mathbf{d}_a$  is given by  $\mathbf{Q}_* = \mathbf{d}_a \otimes \mathbf{h}_{*a} \in \text{Orth}^+$ , from (1) and (19) it follows that

$$\mathbf{F} = \mathbf{F}_* \mathbf{P}, \quad \mathbf{Q} = \mathbf{Q}_* \mathbf{R}. \quad (20)$$

Thus, from (2)<sub>1</sub> and (20) we immediately obtain

$$\mathbf{E}_* = \mathbf{Q}_*^T \mathbf{F}_* - \mathbf{I} = \mathbf{R} \mathbf{E} \mathbf{P}^{-1} + \mathbf{R} \mathbf{P}^{-1} - \mathbf{I} = \mathbf{R}(\mathbf{E} + \mathbf{I})\mathbf{P}^{-1} - \mathbf{I}. \quad (21)$$

To find  $\mathbf{\Gamma}_*$  one needs first express  $\mathbf{B}_*$  through  $\mathbf{B}$ . Using (4), (6)<sub>1</sub>, (19) and applying the chain rule  $\text{Grad}_* \mathbf{h}_{*a} = (\text{Grad} \mathbf{h}_{*a})\mathbf{P}^{-1}$ , we obtain

$$\begin{aligned} \mathbf{B}_* &= \frac{1}{2} \mathbf{h}_{*a} \times \text{Grad}_* \mathbf{h}_{*a} = \frac{1}{2} [(\mathbf{R} \mathbf{h}_a) \text{Grad}(\mathbf{R} \mathbf{h}_a)] \mathbf{P}^{-1} \\ &= \mathbf{R} \left[ \frac{1}{2} \mathbf{h}_a \times \mathbf{R}^T (\mathbf{h}_a \text{Grad} \mathbf{R}^T + \mathbf{R} \text{Grad} \mathbf{h}_a) \right] \mathbf{P}^{-1} \\ &= \mathbf{R} \left[ \frac{1}{2} \mathbf{h}_a \times (\mathbf{h}_a \mathbf{R}^T \text{Grad} \mathbf{R}^T + \text{Grad} \mathbf{h}_a) \right] \mathbf{P}^{-1} \\ &= \mathbf{R} \mathbf{B} \mathbf{P}^{-1} - \mathbf{L}, \end{aligned} \quad (22)$$

$$\begin{aligned} \mathbf{L} &= -\mathbf{R} \left[ \frac{1}{2} \mathbf{h}_a \times (\mathbf{h}_a \mathbf{R} \text{Grad} \mathbf{R}^T) \right] \mathbf{P}^{-1} = -\mathbf{R} \left[ \frac{1}{2} \mathbf{E} : (\mathbf{R} \text{Grad} \mathbf{R}^T) \right] \mathbf{P}^{-1} \\ &= \mathbf{R} \mathbf{Z} \mathbf{P}^{-1}, \quad \mathbf{Z} = -\frac{1}{2} \mathbf{E} : (\mathbf{R} \text{Grad} \mathbf{R}^T). \end{aligned} \quad (23)$$

As a result,

$$\mathbf{\Gamma}_* = \mathbf{Q}_*^T \mathbf{C} \mathbf{F}_* - \mathbf{B}_* = \mathbf{R} \mathbf{Q}^T \mathbf{C} \mathbf{F} \mathbf{P}^{-1} - \mathbf{R} \mathbf{B} \mathbf{P}^{-1} + \mathbf{L} = \mathbf{R} \mathbf{\Gamma} \mathbf{P}^{-1} + \mathbf{L}. \quad (24)$$

Let us note that the 2nd-order tensors  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{\Gamma}$ ,  $\mathbf{M}_N$  and  $\mathbf{K}$  are axial tensors (pseudotensors), not usual (polar) ones. Axial tensors differ from polar ones in that they change signs under inversion transformation  $-\mathbf{I}$  of 3D space  $E$ . Pseudoscalars, pseudovectors and pseudotensors are widely used in modern physics, see for example Nye (1957); Korn and Korn (1968); Feynman et al. (1977); Arfken and Weber (2000). An example of pseudovector is the vector product  $\mathbf{a} \times \mathbf{b}$  of two polar vectors  $\mathbf{a}, \mathbf{b} \in E$ . If  $\mathbf{O} \in Orth$  is an orthogonal tensor then  $(\mathbf{O}\mathbf{a}) \times (\mathbf{O}\mathbf{b}) = (\det \mathbf{O})\mathbf{O}(\mathbf{a} \times \mathbf{b})$  and the vector

product changes sign if  $\mathbf{O} = -\mathbf{I}$ . An example of axial 2nd-order tensor is the skew tensor  $\mathbf{\Omega} = \boldsymbol{\omega} \times \mathbf{I}$  with the polar vector  $\boldsymbol{\omega} \in E$ . An example of axial 3rd-order tensor is the permutation tensor  $\mathbf{E}$ .

Within classical (or Cauchy type) continuum discussed for example in Truesdell and Noll (1965); Truesdell (1966, 1991) it is enough to consider polar 2nd-order tensors when defining the material symmetry group of Green-elastic body. However, for the micropolar (or Cosserat type) continuum one has to take into account that  $\mathbf{\Gamma}$  and  $\mathbf{B}$  appearing in the density  $W_{\mathcal{X}}$  defined in (7) are axial 2nd-order tensors, not the polar ones. Thus one has to extend  $B_*$  by allowing  $\mathbf{R}$  in (7) to be an orthogonal tensor  $\mathbf{R} \in Orth$  and by requiring  $\mathbf{P}$  to be non-singular tensor,  $\det \mathbf{P} \neq 0$ . Additional inversion of space orientation does not correspond to any real deformation of the reference placement. It simply reflects necessary invariance of constitutive equations under mirror reflection of the reference placement or, equivalently, under change of orientation of the base vectors  $\mathbf{h}_a$  from right-hand to left-hand one. Then the refined transformations (22) and (24) become

$$\mathbf{B}_* = (\det \mathbf{R})\mathbf{R}\mathbf{B}\mathbf{P}^{-1} - \mathbf{L}, \quad \mathbf{\Gamma}_* = (\det \mathbf{R})\mathbf{R}\mathbf{\Gamma}\mathbf{P}^{-1} + \mathbf{L}. \quad (25)$$

#### 4. Definition of material symmetry group

The form of elastic strain energy density  $W_{\mathcal{X}}$  of the micropolar body at any particle  $X \in \mathcal{B}$  depends upon the choice of reference placement, in general. Particularly important are sets of reference placements which leave unchanged the form of energy density. Transformations of reference placement under which the energy density remains unchanged are called here invariant transformations. Knowledge of all invariant transformations allows

one to precisely define fluid, solid, liquid crystal or subfluid as well as to introduce notions of isotropic, hemitropic or orthotropic polar-elastic continua, for example. Similar approach is used in classical continuum mechanics and in non-linear elasticity by Truesdell (1964); Truesdell and Noll (1965); Wang and Truesdell (1973); Truesdell (1991); Rivlin (1980), as well as in non-linear theories of shells by Wang (1973); Murdoch and Cohen (1979); Eremeyev and Pietraszkiewicz (2006).

The elastic strain energy density  $W_*$  relative to  $B_*$  depends in each point  $x_* \in B_*$  on the stretch tensor  $\mathbf{E}_*$ , the wryness tensor  $\mathbf{\Gamma}_*$ , and also upon the structure curvature tensor  $\mathbf{B}_*$ . This dependence may, in general, be different than that of  $W_{\varkappa}(\mathbf{E}, \mathbf{\Gamma}; \mathbf{x}, \mathbf{B})$ . However, the strain energy of any part of continuum should be conserved, so that

$$\iiint_{P_{\varkappa}} W_{\varkappa} dv_{\varkappa} = \iiint_{P_*} W_* dv_* \quad (26)$$

for any part  $P_{\varkappa} \subset B_{\varkappa}$  corresponding to  $P_* \subset B_*$ , because the functions  $W_{\varkappa}$  and  $W_*$  describe strain energy densities of the same deformed state of  $P_{\gamma} \subset B_{\gamma} = \chi(P_{\varkappa}) = \chi_*(P_*)$ , where  $\chi_*$  is deformation function from  $B_*$  to  $B_{\gamma}$ .

Changing variables  $\mathbf{x}_* \rightarrow \mathbf{x}$  in the right-hand integral of (26), we obtain

$$\begin{aligned} & \iiint_{P_*} W_*[\mathbf{E}_*(\mathbf{x}_*), \mathbf{\Gamma}_*(\mathbf{x}_*); \mathbf{x}_*, \mathbf{B}_*(\mathbf{x}_*)] dv_* \\ &= \iiint_{P_{\varkappa}} |\det \mathbf{P}| W_*[\mathbf{E}_*(\mathbf{x}), \mathbf{\Gamma}_*(\mathbf{x}); \mathbf{x}, \mathbf{B}_*(\mathbf{x})] dv_{\varkappa}. \end{aligned}$$

Thus, from (26) it follows that  $W_*$  and  $W_{\varkappa}$  are related by

$$|\det \mathbf{P}| W_*(\mathbf{E}_*, \mathbf{\Gamma}_*; \mathbf{x}, \mathbf{B}_*) = W_{\varkappa}(\mathbf{E}, \mathbf{\Gamma}; \mathbf{x}, \mathbf{B}).$$

Here  $\mathbf{E}_*$ ,  $\mathbf{\Gamma}_*$ , and  $\mathbf{B}_*$  are expressed as in (21) and (25).

From physical reasons invariant transformations of the reference placement should preserve elementary volume of  $B_{\mathcal{Z}}$ . Hence, the transformation tensor  $\mathbf{P}$  should belong to the unimodular group for which  $|\det \mathbf{P}| = 1$ .

The assumption that the constitutive relation be insensitive to the change of reference placement means that explicit forms of  $W_{\mathcal{Z}}$  and  $W_*$  should coincide, that is

$$W_{\mathcal{Z}}(\mathbf{E}, \mathbf{\Gamma}; \mathbf{x}, \mathbf{B}) = W_{\mathcal{Z}}(\mathbf{E}_*, \mathbf{K}_*; \mathbf{x}, \mathbf{B}_*).$$

In other words, one may use the same function for the strain energy density independently upon the choice of  $B_{\mathcal{Z}}$  or  $B_*$ , but with different expressions for stretch and wryness tensors as well as for the microstructure curvature tensor. In what follows we not always explicitly indicate that all functions depend also on the position vector  $\mathbf{x}$  and  $W$  is taken relative to the undeformed placement  $B_{\mathcal{Z}}$ .

Using (21) and (25) we obtain the following invariance requirement for  $W$  under change of reference placement:

$$\begin{aligned} & W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) \\ &= W \left[ \mathbf{R}\mathbf{E}\mathbf{P}^{-1} + \mathbf{R}\mathbf{P}^{-1} - \mathbf{I}, (\det \mathbf{R})\mathbf{R}\mathbf{\Gamma}\mathbf{P}^{-1} + \mathbf{L}; (\det \mathbf{R})\mathbf{R}\mathbf{B}\mathbf{P}^{-1} - \mathbf{L} \right]. \end{aligned} \tag{27}$$

The relation (27) holds locally, i.e. it should be satisfied at any  $x$  and  $\mathbf{B}$ , and tensors  $\mathbf{P}$ ,  $\mathbf{R}$ ,  $\mathbf{L}$  are treated as independent here. As a result, local invariance of  $W$  under change of reference placement is described by triple of tensors  $(\mathbf{P}, \mathbf{R}, \mathbf{L})$ .

In what follows we shall use the following nomenclature:

*Orth* =  $\{\mathbf{O} : \mathbf{O}^{-1} = \mathbf{O}^T, \det \mathbf{O} = \pm 1\}$  – the group of orthogonal tensors;



$Orth^+ = \{\mathbf{O} : \mathbf{O} \in Orth, \det \mathbf{O} = 1\}$  – the group of rotation tensors;

$Unim = \{\mathbf{P} : \mathbf{P} \in E \otimes E, \det \mathbf{P} = \pm 1\}$  – the unimodular group;

$Lin = \{\mathbf{L} \in E \otimes E\}$  – the linear group.

Here  $Orth$  and  $Unim$  are groups with regard to multiplication, and  $Lin$  is the group with regard to addition.

Now we are able to introduce the following definition:

**Definition 1.** *By the material symmetry group  $\mathcal{G}_x$  at  $x$  and  $\mathbf{B}$  of the polar-elastic continuum we call all sets of ordered triples of tensors*

$$\mathbb{X} = (\mathbf{P} \in Unim, \mathbf{R} \in Orth, \mathbf{L} \in Lin), \quad (28)$$

*satisfying the relation*

$$\begin{aligned} & W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) \\ & = W [\mathbf{R}\mathbf{E}\mathbf{P}^{-1} + \mathbf{R}\mathbf{P}^{-1} - \mathbf{I}, (\det \mathbf{R})\mathbf{R}\mathbf{\Gamma}\mathbf{P}^{-1} + \mathbf{L}; (\det \mathbf{R})\mathbf{R}\mathbf{B}\mathbf{P}^{-1} - \mathbf{L}] \end{aligned} \quad (29)$$

*for any tensors  $\mathbf{E}, \mathbf{\Gamma}, \mathbf{B}$  in domain of definition of function  $W$ .*

The set  $\mathcal{G}_x$  is group relative to the group operation  $\circ$  defined by

$$(\mathbf{P}_1, \mathbf{R}_1, \mathbf{L}_1) \circ (\mathbf{P}_2, \mathbf{R}_2, \mathbf{L}_2) = [\mathbf{P}_1\mathbf{P}_2, \mathbf{R}_1\mathbf{R}_2, \mathbf{L}_1 + (\det \mathbf{R}_1)\mathbf{R}_1\mathbf{L}_2\mathbf{P}_1^{-1}].$$

Let us check that if  $\mathbb{X}_1 \equiv (\mathbf{P}_1, \mathbf{R}_1, \mathbf{L}_1) \in \mathcal{G}_x$  and  $\mathbb{X}_2 \equiv (\mathbf{P}_2, \mathbf{R}_2, \mathbf{L}_2) \in \mathcal{G}_x$ , then also  $\mathbb{X}_1 \circ \mathbb{X}_2 \in \mathcal{G}_x$ . Indeed, if  $\mathbb{X}_1 \in \mathcal{G}_x$  and  $\mathbb{X}_2 \in \mathcal{G}_x$  then

$$\begin{aligned} & W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) \\ & = W [\mathbf{R}_1\mathbf{E}\mathbf{P}_1^{-1} + \mathbf{R}_1\mathbf{P}_1^{-1} - \mathbf{I}, (\det \mathbf{R}_1)\mathbf{R}_1\mathbf{\Gamma}\mathbf{P}_1^{-1} + \mathbf{L}_1; (\det \mathbf{R}_1)\mathbf{R}_1\mathbf{B}\mathbf{P}_1^{-1} - \mathbf{L}_1] \\ & = W [\mathbf{R}_2\mathbf{E}\mathbf{P}_2^{-1} + \mathbf{R}_2\mathbf{P}_2^{-1} - \mathbf{I}, (\det \mathbf{R}_2)\mathbf{R}_2\mathbf{\Gamma}\mathbf{P}_2^{-1} + \mathbf{L}_2; (\det \mathbf{R}_2)\mathbf{R}_2\mathbf{B}\mathbf{P}_2^{-1} - \mathbf{L}_2]. \end{aligned}$$

Taking these relations into account we have

$$\begin{aligned}
& W \left[ \mathbf{R}_1 \mathbf{R}_2 \mathbf{E} \mathbf{P}_2^{-1} \mathbf{P}_1^{-1} + \mathbf{R}_1 \mathbf{R}_2 \mathbf{P}_2^{-1} \mathbf{P}_1^{-1} - \mathbf{I}, \right. \\
& \quad (\det \mathbf{R}_1)(\det \mathbf{R}_2) \mathbf{R}_1 \mathbf{R}_2 \mathbf{\Gamma} \mathbf{P}_2^{-1} \mathbf{P}_1^{-1} + \mathbf{L}_1 + (\det \mathbf{R}_1) \mathbf{R}_1 \mathbf{L}_2 \mathbf{P}_1^{-1}; \\
& \quad \left. (\det \mathbf{R}_1)(\det \mathbf{R}_2) \mathbf{R}_1 \mathbf{R}_2 \mathbf{B} \mathbf{P}_2^{-1} \mathbf{P}_1^{-1} - \mathbf{L}_1 - (\det \mathbf{R}_1) \mathbf{R}_1 \mathbf{L}_2 \mathbf{P}_1^{-1} \right] \\
&= W \left\{ \mathbf{R}_1 \left( \mathbf{R}_2 \mathbf{E} \mathbf{P}_2^{-1} + \mathbf{R}_2 \mathbf{P}_2^{-1} - \mathbf{I} \right) \mathbf{P}_1^{-1} + \mathbf{R}_1 \mathbf{P}_1^{-1} - \mathbf{I}, \right. \\
& \quad (\det \mathbf{R}_1) \mathbf{R}_1 \left[ (\det \mathbf{R}_2) \mathbf{R}_2 \mathbf{\Gamma} \mathbf{P}_2^{-1} + \mathbf{L}_2 \right] \mathbf{P}_1^{-1} + \mathbf{L}_1; \\
& \quad \left. (\det \mathbf{R}_1) \mathbf{R}_1 \left[ (\det \mathbf{R}_2) \mathbf{R}_2 \mathbf{B} \mathbf{P}_2^{-1} - \mathbf{L}_2 \right] \mathbf{P}_1^{-1} - \mathbf{L}_1 \right\} \\
&= W \left[ \mathbf{R}_2 \mathbf{E} \mathbf{P}_2^{-1} + \mathbf{R}_2 \mathbf{P}_2^{-1} - \mathbf{I}, \right. \\
& \quad \left. (\det \mathbf{R}_2) \mathbf{R}_2 \mathbf{\Gamma} \mathbf{P}_2^{-1} + \mathbf{L}_2; (\det \mathbf{R}_2) \mathbf{R}_2 \mathbf{B} \mathbf{P}_2^{-1} - \mathbf{L}_2 \right] \\
&= W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}),
\end{aligned}$$

which proves that  $\mathbb{X}_1 \circ \mathbb{X}_2$  belongs to the group  $\mathcal{G}_\varkappa$  indeed.

The unit element of  $\mathcal{G}_\varkappa$  is  $\mathbb{I} = (\mathbf{I}, \mathbf{I}, \mathbf{0})$ , and the inverse element to  $\mathbb{X} \in \mathcal{G}_\varkappa$  is given by

$$\mathbb{X}^{-1} \equiv (\mathbf{P}, \mathbf{R}, \mathbf{L})^{-1} = [\mathbf{P}^{-1}, \mathbf{R}^T, -(\det \mathbf{R}) \mathbf{R}^T \mathbf{L} \mathbf{P}].$$

Indeed,

$$\begin{aligned}
\mathbb{X} \circ \mathbb{X}^{-1} &\equiv (\mathbf{P}, \mathbf{R}, \mathbf{L}) \circ (\mathbf{P}, \mathbf{R}, \mathbf{L})^{-1} \\
&= [\mathbf{P} \mathbf{P}^{-1}, \mathbf{R} \mathbf{R}^T, \mathbf{L} - (\det \mathbf{R})^2 \mathbf{R} \mathbf{R}^T \mathbf{L} \mathbf{P} \mathbf{P}^{-1}] = (\mathbf{I}, \mathbf{I}, \mathbf{0}).
\end{aligned}$$

Our symmetry group  $\mathcal{G}_\varkappa$  differs from the one of micropolar elastic materials proposed by Eringen and Kafadar (1976). We take into account explicit dependence of  $W$  upon  $\mathbf{B}$  as parametric tensor, while in Eringen and Kafadar (1976) dependence of  $\mathbf{T}$ ,  $\mathbf{M}$  and other quantities on the microinertia tensor  $\mathbf{J}_\varkappa$  is assumed. In our notation, definition of  $\mathcal{G}_\varkappa$  by Eringen and Kafadar (1976) is based on the formula

$$W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{J}_\varkappa) = W[\mathbf{R} \mathbf{E} \mathbf{P}^{-1} + \mathbf{R} \mathbf{P}^{-1} - \mathbf{I}, \mathbf{R} \mathbf{\Gamma} \mathbf{P}^{-1} + \mathbf{L}; \mathbf{R} \mathbf{J}_\varkappa \mathbf{R}^T]. \quad (30)$$

The microinertia tensor  $\mathbf{J}_x$  appearing in (30) has another nature and transforms by another rule than  $\mathbf{B}$  under change of reference placement. Moreover, in what follows we take into account that  $\mathbf{\Gamma}$  and  $\mathbf{B}$  as arguments of  $W$  are axial tensors. Eringen and Kafadar (1976) did not take into account that  $\mathbf{\Gamma}$  was the axial tensor. The mentioned differences in definitions of  $\mathcal{G}_x$  lead to different restrictions of constitutive relations of polar-elastic continua.

Let us show that Definition 1 allows one to establish an analogue of Noll's rule given for classical simple material continuum by Noll (1958). Since the material symmetry group depends not only on particle  $X \in \mathcal{B}$  but also upon choice of reference placement, let us analyse how symmetry groups corresponding to different reference placements are related. Let  $B_1$  and  $B_2$  be two different reference placements, and  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be material symmetry groups relative to these reference placements, respectively. In what follows quantities described in the placements  $B_1$  and  $B_2$  are marked by the respective lower indices 1 and 2.

Let now  $\mathbf{P}$  be a non-singular deformation gradient,  $\det \mathbf{P} \neq 0$ ,  $\mathbf{R}$  be an orthogonal tensor associated with the transformation  $B_1 \rightarrow B_2$ , as well as  $\mathbf{P}^{-1}$  and  $\mathbf{R}^T$  be an inverse deformation gradient and an inverse orthogonal tensor associated with an inverse transformation  $B_2 \rightarrow B_1$ , respectively. Cases  $\det \mathbf{P} < 1$  and  $\det \mathbf{R} = -1$  correspond to change from right-handed frame in  $B_1$  to left-handed one in  $B_2$ . Then, by analogy to (21) and (25) we can relate the strain measures  $\mathbf{E}_1$  and  $\mathbf{E}_2$ ,  $\mathbf{\Gamma}_1$  and  $\mathbf{\Gamma}_2$ , as well as the structure curvature tensors  $\mathbf{B}_1$  and  $\mathbf{B}_2$  defined relative to two different reference placements by

$$\mathbf{E}_2 = \mathbf{R}\mathbf{E}_1\mathbf{P}^{-1} + \mathbf{R}\mathbf{P}^{-1} - \mathbf{I}, \quad \mathbf{\Gamma}_2 = (\det \mathbf{R})\mathbf{R}\mathbf{\Gamma}_1\mathbf{P}^{-1} + \mathbf{L}. \quad (31)$$

$$\mathbf{B}_2 = (\det \mathbf{R})\mathbf{R}\mathbf{B}_1\mathbf{P}^{-1} - \mathbf{L}, \quad \mathbf{L} = (\det \mathbf{R})\mathbf{R}\mathbf{Z}\mathbf{P}^{-1}. \quad (32)$$

Let  $W_1$  and  $W_2$  be strain energy densities defined relative to the two reference placements. From (26) it follows that  $W_2$  and  $W_1$  are related by

$$|\det \mathbf{P}| W_2(\mathbf{E}_2, \mathbf{\Gamma}_2; \mathbf{B}_2) = W_1(\mathbf{E}_1, \mathbf{\Gamma}_1; \mathbf{B}_1).$$

Taking into account (31) and (32) we have

$$\begin{aligned} & |\det \mathbf{P}| W_2 [\mathbf{R}\mathbf{E}_1\mathbf{P}^{-1} + \mathbf{R}\mathbf{P}^{-1} - \mathbf{I}, (\det \mathbf{R})\mathbf{R}\mathbf{\Gamma}_1\mathbf{P}^{-1} + \mathbf{L}; (\det \mathbf{R})\mathbf{R}\mathbf{B}_1\mathbf{P}^{-1} - \mathbf{L}] \\ & = W_1(\mathbf{E}_1, \mathbf{\Gamma}_1; \mathbf{B}_1). \end{aligned} \tag{33}$$

Let the element  $\mathbb{X}_1 \equiv (\mathbf{P}_1, \mathbf{R}_1, \mathbf{L}_1) \in \mathcal{G}_1$ . Then using (33) we obtain

$$\begin{aligned} & |\det \mathbf{P}| W_2(\mathbf{E}_2, \mathbf{\Gamma}_2; \mathbf{B}_2) \\ & = W_1(\mathbf{E}_1, \mathbf{\Gamma}_1; \mathbf{B}_1) \\ & = W_1 [\mathbf{R}_1\mathbf{E}_1\mathbf{P}_1^{-1} + \mathbf{R}_1\mathbf{I}\mathbf{P}_1^{-1} - \mathbf{I}, (\det \mathbf{R}_1)\mathbf{R}_1\mathbf{\Gamma}_1\mathbf{P}_1^{-1} + \mathbf{L}_1; \\ & \quad (\det \mathbf{R}_1)\mathbf{R}_1\mathbf{B}_1\mathbf{P}_1^{-1} - \mathbf{L}_1] \\ & = |\det \mathbf{P}| W_2 [\mathbf{R}\mathbf{R}_1\mathbf{E}_1\mathbf{P}_1^{-1}\mathbf{P}^{-1} + \mathbf{R}\mathbf{R}_1\mathbf{I}\mathbf{P}_1^{-1}\mathbf{P}^{-1} - \mathbf{I}, \\ & \quad (\det \mathbf{R})(\det \mathbf{R}_1)\mathbf{R}\mathbf{R}_1\mathbf{K}_1\mathbf{P}_1^{-1}\mathbf{P}^{-1} \\ & \quad + (\det \mathbf{R})\mathbf{R}\mathbf{L}_1\mathbf{P}^{-1} + \mathbf{L}; \\ & \quad (\det \mathbf{R})(\det \mathbf{R}_1)\mathbf{R}\mathbf{R}_1\mathbf{B}_1\mathbf{P}_1^{-1}\mathbf{P}^{-1} \\ & \quad - (\det \mathbf{R})\mathbf{R}\mathbf{L}_1\mathbf{P}^{-1} - \mathbf{L}] \\ & = |\det \mathbf{P}| W_2 [\mathbf{R}\mathbf{R}_1\mathbf{R}^T\mathbf{E}_2\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1} + \mathbf{R}\mathbf{R}_1\mathbf{R}^T\mathbf{I}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1} - \mathbf{I}, \\ & \quad (\det \mathbf{R}_1)\mathbf{R}\mathbf{R}_1\mathbf{R}^T\mathbf{K}_2\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1} \\ & \quad - (\det \mathbf{R}_1)\mathbf{R}\mathbf{R}_1\mathbf{R}^T\mathbf{L}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1} \\ & \quad + (\det \mathbf{R}_1)\mathbf{R}\mathbf{L}_1\mathbf{P}^{-1} + \mathbf{L}; \\ & \quad (\det \mathbf{R}_1)\mathbf{R}\mathbf{R}_1\mathbf{R}^T\mathbf{B}_2\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1} \\ & \quad + (\det \mathbf{R}_1)\mathbf{R}\mathbf{R}_1\mathbf{R}^T\mathbf{L}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1} \\ & \quad - (\det \mathbf{R}_1)\mathbf{R}\mathbf{L}_1\mathbf{P}^{-1} - \mathbf{L}]. \end{aligned} \tag{34}$$

From (34) it follows that the element  $\mathbb{X}_2 \equiv (\mathbf{P}_2, \mathbf{R}_2, \mathbf{L}_2) \in \mathcal{G}_2$ , where

$$\begin{aligned}\mathbf{P}_2 &= \mathbf{P}\mathbf{P}_1\mathbf{P}^{-1}, & \mathbf{R}_2 &= \mathbf{R}\mathbf{R}_1\mathbf{R}^T, \\ \mathbf{L}_2 &= \mathbf{L} + (\det \mathbf{R}_1)\mathbf{R}\mathbf{L}_1\mathbf{P}^{-1} - (\det \mathbf{R}_1)\mathbf{R}\mathbf{R}_1\mathbf{R}^T\mathbf{L}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1}.\end{aligned}$$

It is easy to show that  $\mathbb{X}_2 = \mathbb{P} \circ \mathbb{X}_1 \circ \mathbb{P}^{-1}$ , where  $\mathbb{P} \equiv (\mathbf{P}, \mathbf{R}, \mathbf{L})$ . Indeed,

$$\begin{aligned}\mathbb{P} \circ \mathbb{X}_1 &\equiv (\mathbf{P}, \mathbf{R}, \mathbf{L}) \circ (\mathbf{P}_1, \mathbf{R}_1, \mathbf{L}_1) \\ &= [\mathbf{P}\mathbf{P}_1, \mathbf{R}\mathbf{R}_1, \mathbf{L} + (\det \mathbf{R})\mathbf{R}^T\mathbf{L}\mathbf{P}].\end{aligned}$$

Taking into account that  $\mathbb{P}^{-1} = [\mathbf{P}^{-1}, \mathbf{R}^T, -(\det \mathbf{R})\mathbf{R}^T\mathbf{L}\mathbf{P}]$ , we obtain

$$\begin{aligned}\mathbb{P} \circ \mathbb{X}_1 \circ \mathbb{P}^{-1} &= [\mathbf{P}\mathbf{P}_1\mathbf{P}^{-1}, \mathbf{R}\mathbf{R}_1\mathbf{R}^T, \mathbf{L} + (\det \mathbf{R})^2(\det \mathbf{R}_1)\mathbf{R}\mathbf{L}_1\mathbf{P}^{-1} \\ &\quad - (\det \mathbf{R})^2(\det \mathbf{R}_1)\mathbf{R}\mathbf{R}_1\mathbf{R}^T\mathbf{L}\mathbf{P}\mathbf{P}_1^{-1}\mathbf{P}^{-1}],\end{aligned}$$

from which follows the sought result.

Thus the material symmetry group under change of reference placement transforms according to

$$\mathcal{G}_2 = \mathbb{P} \circ \mathcal{G}_1 \circ \mathbb{P}^{-1}. \quad (35)$$

The transformation (35) is a counterpart in non-linear micropolar elasticity of the well known Noll rule for symmetry groups of simple materials in classical continuum mechanics, see Noll (1958); Truesdell and Noll (1965); Wang and Truesdell (1973).

## 5. Material symmetry and reduced forms of strain energy density

The structure of  $\mathcal{G}_\varkappa$  puts some constraints on the form of  $W$  which allow us to considerably simplify this form.

In this Section we consider some simple specific cases of  $\mathcal{G}_\varkappa$ .

**Case 1.** Let us discuss the trivial symmetry group  $\mathcal{G}_\varkappa$  consisting of two elements:  $\mathbb{I}$  and  $-\mathbb{I} = (-\mathbf{I}, -\mathbf{I}, \mathbf{0})$ . Applying  $-\mathbb{I}$ , from definition of  $\mathcal{G}_\varkappa$  it follows that  $W$  should be an even function of  $\mathbf{\Gamma}$  and  $\mathbf{B}$ , that is

$$W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) = W(\mathbf{E}, -\mathbf{\Gamma}; -\mathbf{B}). \quad (36)$$

The condition (36) means, in particular, that explicit expression of  $W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B})$  cannot have terms linear in  $\mathbf{\Gamma}$  alone such as  $\text{tr}(\mathbf{E}^T \mathbf{\Gamma})$ . Only when explicit dependence of  $W$  on  $\mathbf{B}$  is taken into account, terms linear in  $\mathbf{\Gamma}$  of the type  $\text{tr}(\mathbf{B}^T \mathbf{\Gamma})$  or  $\text{tr}(\mathbf{B}^T \mathbf{\Gamma} \mathbf{E})$  are allowed.

**Case 2.** If  $\mathcal{G}_\varkappa$  consists of tensor triples containing two identities and an arbitrary tensor  $\mathbf{L} \in Lin$ , then the number of arguments in  $W$  can be reduced.

Indeed, let  $\mathbb{X} = (\mathbf{I}, \mathbf{I}, \mathbf{L}) \in \mathcal{G}_\varkappa$ . Then

$$W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) = W(\mathbf{E}, \mathbf{\Gamma} + \mathbf{L}; \mathbf{B} - \mathbf{L}), \quad \forall \mathbf{L} \in Lin. \quad (37)$$

Introducing one-parameter family of transformations

$$W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) = W(\mathbf{E}, \mathbf{\Gamma} + t\mathbf{L}; \mathbf{B} - t\mathbf{L}), \quad \forall \mathbf{L} \in Lin, \forall t \in R,$$

and differentiating the latter equation with regard to  $t$  at  $t = 0$ , we find that

$$0 = \frac{\partial W_\varkappa}{\partial \mathbf{\Gamma}} : \mathbf{L} - \frac{\partial W_\varkappa}{\partial \mathbf{B}} : \mathbf{L}, \quad \forall \mathbf{L} \in Lin.$$

This equation is satisfied if and only if

$$W = W(\mathbf{E}, \mathbf{\Gamma} + \mathbf{B}) = \widetilde{W}(\mathbf{E}, \mathbf{\Pi}),$$

where  $\mathbf{\Pi} = \mathbf{\Gamma} + \mathbf{B}$ . Here  $:$  denotes the double-dot (scalar) product of two 2nd-order tensors,  $\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^T \mathbf{B}) = A_{mn} B_{mn}$ .

Alternatively, since (37) should be satisfied by any  $\mathbf{L}$  let us take  $\mathbf{L}$  to be equal  $\mathbf{B}$ . Then from (37) it directly follows that

$$W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) = W(\mathbf{E}, \mathbf{\Gamma} + \mathbf{B}; \mathbf{0}) = \widetilde{W}(\mathbf{E}, \mathbf{\Pi}).$$

**Case 3.** Let us consider the reduced form  $W = W(\mathbf{E}, \mathbf{\Gamma})$ , i.e. let us neglect its explicit dependence on  $\mathbf{B}$ . This form is widely used in the literature, see for example Maugin (1998); Nikitin and Zubov (1998); Eringen (1999); Ramezani and Naghdabadi (2007); Ramezani et al. (2009). If  $\mathcal{G}_\times = \{\mathbb{X} : \mathbb{X} = (\mathbf{I}, \mathbf{I}, \mathbf{L})\}$  contains an arbitrary tensor  $\mathbf{L} \in Lin$ , we have  $W(\mathbf{E}, \mathbf{\Gamma}) = W(\mathbf{E}, \mathbf{\Gamma} + \mathbf{L})$ . Then we can again introduce one-parameter family of transformations

$$W(\mathbf{E}, \mathbf{\Gamma}) = W(\mathbf{E}, \mathbf{\Gamma} + t\mathbf{L}), \quad \forall \mathbf{L} \in Lin, \forall t \in R.$$

Differentiating this equation with regard to  $t$  at  $t = 0$ , we obtain

$$\frac{\partial W}{\partial \mathbf{\Gamma}} : \mathbf{L} = \mathbf{0}, \quad \forall \mathbf{L} \in Lin,$$

from which it follows that  $\frac{\partial W}{\partial \mathbf{\Gamma}} = \mathbf{0}$ . Thus in this case  $W$  takes the reduced form

$$W = W(\mathbf{E}). \tag{38}$$

Such a significant reduction of  $W = W(\mathbf{E}, \mathbf{\Gamma})$  follows directly from assumption that the material symmetry group contains an arbitrary tensor  $\mathbf{L}$ . To keep the second argument  $\mathbf{\Gamma}$  in  $W$  it is necessary to avoid using material symmetry groups containing an arbitrary tensor  $\mathbf{L}$ .

The form (38) of  $W$  corresponds to so-called reduced elastic Cosserat continuum in which the couple-stress tensor  $\mathbf{K}$  vanishes while the stress tensor  $\mathbf{S}$  still remains non-symmetric, in general. The couple equilibrium equations (10) and (15)<sub>2</sub> reduce to

$$\text{ax}(\mathbf{T}_N^T \mathbf{F}^T - \mathbf{F} \mathbf{T}_N) + \rho_\varkappa \mathbf{c} = \mathbf{0} \quad \text{in } B_\varkappa, \quad (39)$$

$$\text{ax}(\mathbf{T}^T - \mathbf{T}) + \rho \mathbf{c} = \mathbf{0} \quad \text{in } B_\gamma. \quad (40)$$

These equations become non-trivial here and can be used to find the field of rotation  $\mathbf{Q}$ . However, since (39) and (40) constitute the system of non-linear algebraic equations with respect to  $\mathbf{Q}$ , it is not possible to assume any rotation  $\mathbf{Q}$  at the boundary  $\partial B_{\varkappa d}$ .

The linear version of reduced Cosserat elastic continuum was used to describe wave propagation in soils and rocks, see for example Grekova et al. (2009).

**Case 4.** Let the material symmetry group of reduced Cosserat continuum (Case 3) be wider:  $\mathcal{G}_\varkappa = \{\mathbb{X} : \mathbb{X} = (\mathbf{I}, \mathbf{R} \in Orth^+, \mathbf{L} \in Lin)\}$ . Then it describes the classical non-linear Green-elastic material (Truesdell and Noll, 1965; Ogden, 1984) for which  $W = W(\mathbf{U})$ , where  $\mathbf{U} = (\mathbf{F}^T \mathbf{F})^{1/2}$ .

Within the reduced Cosserat continuum tensors  $\mathbf{P}$ ,  $\mathbf{R}$  and  $\mathbf{L}$  belong to  $\mathcal{G}_\varkappa$  if

$$W(\mathbf{E}) = W(\mathbf{R} \mathbf{E} \mathbf{P}^{-1} + \mathbf{R} \mathbf{P}^{-1} - \mathbf{I}) = W(\mathbf{R} \mathbf{Q}^T \mathbf{F} \mathbf{P}^{-1} - \mathbf{I}).$$

Let us substitute here  $\mathbf{P} = \mathbf{I}$  and  $\mathbf{R} = \mathbf{A}^T \mathbf{Q}$ , where  $\mathbf{A}$  is the proper orthogonal tensor following from the polar decomposition  $\mathbf{F} = \mathbf{A} \mathbf{U}$  of deformation gradient  $\mathbf{F}$  and  $\mathbf{Q} \in Orth^+$ . Then we obtain

$$W(\mathbf{E}) = W(\mathbf{A}^T \mathbf{Q} \mathbf{Q}^T \mathbf{F} - \mathbf{I}) = W(\mathbf{A}^T \mathbf{F} - \mathbf{I}) = W(\mathbf{U} - \mathbf{I}) = W^\times(\mathbf{U}).$$



In this case the corresponding Cauchy stress tensor  $\mathbf{T}$  becomes symmetric,  $\mathbf{T} = \mathbf{T}^T$ . The proof is based on straightforward calculation of  $\mathbf{S}$  using  $W = W^\times(\mathbf{U})$  which gives us the formula

$$\mathbf{T} = (\det \mathbf{F})^{-1} \mathbf{A} \mathbf{U} \mathbf{S}^\times \mathbf{A}^T,$$

where  $\mathbf{S}^\times = dW^\times/d\mathbf{U}$ , and  $\mathbf{S}^{\times T} = \mathbf{S}^\times$ .

Since  $\mathbf{K} = \mathbf{0}$ , the couple equilibrium equations (10) or (15)<sub>2</sub> are identically satisfied for vanishing couple vector  $\mathbf{c}$ . Material symmetry groups and representations of constitutive equations of non-linear elastic continuum were discussed for example in Truesdell and Noll (1965); Wang and Truesdell (1973); Ogden (1984).

Four simple specific cases discussed above demonstrate that knowledge of elements of the material symmetry group allows one to substantially simplify the form of  $W$ . In next sections we consider some cases of  $\mathcal{G}_x$  which lead to definitions of polar-elastic solids, fluids, liquid crystals, or subfluids, and to their reduced constitutive equations.

## 6. Polar-elastic isotropic material

As in case of non-polar elastic materials, property of isotropy of the polar-elastic material is expressed in terms of orthogonal group.

**Definition 2.** *The micropolar elastic continuum is called isotropic at  $x$  and  $\mathbf{B}$  if there exists a reference placement  $B_x$ , called undistorted, such that the material symmetry group relative to  $B_x$  contains the group  $\mathcal{I}_x$ ,*

$$\mathcal{I}_x \subset \mathcal{G}_x, \quad \mathcal{I}_x \equiv \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0}) : \mathbf{O} \in Orth\}. \quad (41)$$

From physical point of view this definition means that uniform rotations and mirror reflections of the undistorted reference placement  $B_x$  cannot be recognized by any experiment.

## 7. Polar-elastic fluids

To strain energy density of micropolar elastic fluids  $w$  should be insensitive to any change of reference placement, i.e. the equation (27) should be satisfied by any triple of tensors  $\mathbf{P} \in Unim$ ,  $\mathbf{R} \in Orth$ ,  $\mathbf{L} \in Lin$ .

**Definition 3.** *The micropolar elastic continuum is called the polar-elastic fluid at  $x$  and  $\mathbf{B}$  if there exists a reference placement  $B_x$ , called undistorted, such that the material symmetry group relative to  $B_x$  is given by*

$$\mathcal{G}_\kappa = \mathcal{U}_\kappa \equiv \{(\mathbf{P} \in Unim, \mathbf{R} \in Orth, \mathbf{L} \in Lin)\}. \quad (42)$$

Hence, the strain energy density of the polar-elastic fluid satisfies the relation

$$\begin{aligned} & W_x(\mathbf{E}, \mathbf{F}; \mathbf{B}) \\ &= W_x[\mathbf{R}\mathbf{E}\mathbf{P}^{-1} + \mathbf{R}\mathbf{P}^{-1} - \mathbf{I}, (\det \mathbf{R})\mathbf{R}\mathbf{F}\mathbf{P}^{-1} + \mathbf{L}; (\det \mathbf{R})\mathbf{R}\mathbf{B}\mathbf{P}^{-1} - \mathbf{L}], \\ & \forall \mathbf{P} \in Unim, \quad \forall \mathbf{R} \in Orth, \quad \forall \mathbf{L} \in Lin. \end{aligned} \quad (43)$$

From Noll's rule (35) it is easy to verify that any reference placement of the polar-elastic fluid becomes undistorted, similarly as it is for the non-polar elastic fluid, because the symmetry group becomes here maximal. Obviously, any polar-elastic fluid is also isotropic.

Strain energy density of polar-elastic fluid satisfying (43) takes the form

$$W = W(\mathbf{E}, \mathbf{\Pi}) = W^\times(\det \mathbf{F}, \mathbf{C}), \quad (44)$$

where  $\mathbf{C}$  is the structure curvature tensor of the deformed placement  $B_\gamma$  defined in (6)<sub>2</sub>. Since  $\mathbf{C}$  does not depend on the choice of reference placements, it is easy to check that (44) satisfies the requirement (43).

The strain energy density (44) describes polar-elastic continuum which is insensitive to arbitrary deformations preserving an elementary volume element. However, it is still sensitive to change of orientation of particles.

By the principle of material frame-indifference (invariance under superposed rigid-body deformation) the function  $W^\times$  satisfies the condition

$$W(\det \mathbf{F}, \mathbf{C}) = W^\times[\det(\mathbf{Q}^T \mathbf{F}), \mathbf{O}^T \mathbf{C} \mathbf{O}], \quad \forall \mathbf{O} \in Orth^+,$$

i.e.  $W^\times$  is the hemitropic function with regard to  $\mathbf{C}$ .

Using general representations of isotropic and hemitropic scalar-valued functions of one non-symmetric tensor  $\mathbf{C}$  given by Spencer and Rivlin (1962); Smith (1965, 1994); Spencer (1965, 1971); Smith and Smith (1971) with the help of algebraic theory of the invariants,  $W^\times$  can be constructed as a function of six invariants  $j_n$ ,  $n = 1, \dots, 6$ ,

$$W = W^\times(\det \mathbf{F}, j_1, j_2, \dots, j_6) \tag{45}$$

where

$$\begin{aligned} j_1 &= \text{tr } \mathbf{C}, & j_2 &= \text{tr } \mathbf{C}^2, & j_3 &= \text{tr } \mathbf{C}^3, & j_4 &= \text{tr } (\mathbf{C} \mathbf{C}^T), \\ j_5 &= \text{tr } (\mathbf{C}^2 \mathbf{C}^T), & j_6 &= \text{tr } (\mathbf{C}^2 \mathbf{C}^{T2}) \end{aligned} \tag{46}$$

as was used for example in Kafadar and Eringen (1971).

Since  $\det \mathbf{F} = \rho_\varkappa / \rho$ , the strain energy density of the polar-elastic fluid may be given in another form more convenient in hydrodynamics,

$$W = \overline{W}(\rho, \mathbf{C}) = \overline{W}(\rho, j_1, j_2, \dots, j_6). \tag{47}$$

Because  $\mathbf{C}$  is the axial tensor, it changes sign under change of frame orientation. Thus,  $\overline{W}$  should be an even function of  $\mathbf{C}$ ,  $\overline{W}(\rho, \mathbf{C}) = \overline{W}(\rho, -\mathbf{C})$ , and its invariant expression should have the property

$$\overline{W}(\rho, j_1, j_2, j_3, j_4, j_5, j_6) = \overline{W}(\rho, -j_1, j_2, -j_3, j_4, -j_5, j_6).$$

As a simple example of the polynomial strain energy density we can propose the following quadratic function:

$$\overline{W}(\rho, \mathbf{C}) = \alpha_0(\rho) + \alpha_1 j_1^2 + \alpha_2 j_2 + \alpha_3 j_4, \quad (48)$$

where  $\alpha_a$ ,  $a = 1, 2, 3$ , are assumed to be constants.

Constitutive equations for  $\mathbf{T}$  and  $\mathbf{M}$  corresponding to (47) are

$$\mathbf{T} = -p\mathbf{I} - \mathbf{M}\mathbf{C}, \quad \mathbf{M} = \frac{\rho}{\rho_\varkappa} \frac{\partial \overline{W}}{\partial \mathbf{C}^T}, \quad p = \frac{\rho^2}{\rho_\varkappa} \frac{\partial \overline{W}}{\partial \rho}. \quad (49)$$

For example, with the strain energy density (48) the couple-stress tensor  $\mathbf{M}$  is given by the constitutive equation

$$\mathbf{M} = 2 \frac{\rho}{\rho_\varkappa} (\alpha_1 j_1 \mathbf{I} + \alpha_2 \mathbf{C}^T + \alpha_3 \mathbf{C}).$$

Basic equations of viscous micropolar fluids were proposed by Aero et al. (1965); Eringen (1966). Similar theory of fluids with three rigid directors as primary quantities was introduced by Allen et al. (1967). Within the framework of these models the strain energy density had the form  $W = W(\rho)$ , which is the particular case of (47). Strain energy density (44) of the polar-elastic fluids was established by Zubov and Eremeev (1996); Yeremeyev and Zubov (1999) without introduction of material symmetry group.

## 8. Polar-elastic solids

Material symmetry group of non-polar non-linear elastic solids is constructed with the help of orthogonal transformations describing rotations and reflections of reference placement, see Truesdell and Noll (1965); Wang and Truesdell (1973); Ogden (1984). But for polar-elastic continua material symmetry group consists of ordered triple of independent tensors,  $\mathcal{G}_z = (\mathbf{P}, \mathbf{R}, \mathbf{L})$ .

In Definition 3 of the polar-elastic fluid tensors  $\mathbf{P}$  and  $\mathbf{R}$  are entirely independent from each other. Considering directors  $\mathbf{h}_a$  as unit orthogonal vectors rigidly attached to rigid particles (e.g. molecules) of the fluid, it is apparent that  $\mathbf{P}$  need not coincide with  $\mathbf{R}$ . This is also true in the mechanical modelling of sand, powder, etc.

In polar-elastic solids both  $\mathbf{P}$  and  $\mathbf{R}$  describing change of the reference placement should be orthogonal. Then corresponding constitutive equations might be sensitive to differences of both orthogonal tensors  $\mathbf{P}$  and  $\mathbf{R}$ , which would be difficult to accept. Hence, we assume here that  $\mathbf{P} = \mathbf{R} \in Orth$ .

The case  $\mathbf{L} \neq \mathbf{0}$  corresponds to a non-homogeneous field of  $\mathbf{R}$ , see (23). It can be shown that the material symmetry group with  $\mathbf{L} \neq \mathbf{0}$  would lead to constitutive equations describing polar-elastic liquid crystals and subfluids. Hence, for the polar-elastic solids the following hypothesis seems to be physically justified:

**Hypothesis.** *The material symmetry group of the polar-elastic solids consists of all transformations of reference placement performed by the same orthogonal transformations of position vector and directors. Additionally we assume that  $\mathbf{L} = \mathbf{0}$ .*

The hypothesis requires that  $\mathbf{P} = \mathbf{R}$  and values of  $\mathbf{R}$  should belong to a

subgroup of *Orth*.

Accepting the hypothesis we can propose the following definition:

**Definition 4.** *The micropolar elastic continuum is called the polar-elastic solid at  $x$  and  $\mathbf{B}$  if there exists a reference placement  $B_x$ , called undistorted, such that the material symmetry group relative to  $B_x$  is given by*

$$\mathcal{G}_x = \mathcal{R}_x \equiv \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0}) : \mathbf{O} \in \mathcal{O}_x \subset Orth\}. \quad (50)$$

The group  $\mathcal{R}_x$  is fully described by a subgroup  $\mathcal{O}_x$  of orthogonal group *Orth*. Invariance requirement of  $W$  leads here to finding the subgroup  $\mathcal{O}_x$  such that

$$W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) = W[\mathbf{OEO}^T, (\det \mathbf{O})\mathbf{O}\mathbf{F}\mathbf{O}^T; (\det \mathbf{O})\mathbf{O}\mathbf{B}\mathbf{O}^T], \quad \forall \mathbf{O} \in \mathcal{O}_x. \quad (51)$$

Both our definitions of fluids (43) and solids (51) differ from definitions proposed by Eringen and Kafadar (1976) by factors  $\det \mathbf{R}$  and  $\det \mathbf{O}$ , which take into account that our  $\mathbf{\Gamma}$  and  $\mathbf{B}$  are axial tensors. Additionally, the tensor  $\mathbf{B}$  appears instead of  $\mathbf{J}_x$  in the list of arguments of our  $W$ .

## 9. Polar-elastic liquid crystals and subfluids

The strain energy density of polar-elastic continuum may also admit other material symmetry groups, in general. For example, it is possible to construct material symmetry groups of  $W$  in analogy to symmetry groups used to model liquid crystals or subfluids in continuum mechanics of simple materials, see Coleman (1965); Wang (1965); Truesdell and Noll (1965); Wang and Truesdell (1973). However, those mathematical models differ from physical

models of liquid crystals used for nematics, smectics and other materials, which are usually based on introducing one rigid director, c.f. Truesdell and Noll (1965); de Gennes and Prost (1993); Ericksen (1998).

**Definition 5.** *The micropolar elastic continuum is called the polar-elastic liquid crystal at  $x$  and  $\mathbf{B}$  if the material symmetry group  $\mathcal{G}_x$  does not coincide with  $\mathcal{U}_x$ , but there exist elements  $\mathbb{X} \in \mathcal{G}_x$ , which are not members of any group constructed using only orthogonal tensors.*

Definition 5 means that a polar-elastic liquid crystal is neither a polar-elastic fluid nor a polar-elastic solid.

Within micropolar elastic continuum the number of material symmetry groups corresponding to polar-elastic liquid crystals is larger than in case of non-polar elastic liquid crystals, because the structure of  $\mathcal{G}_x$  of the former is more complex. The Cases 2–4 considered above belong to polar-elastic liquid crystals by their definitions. Below we give other examples of polar-elastic liquid crystals.

Grekova and Zhilin (2001) defined the Kelvin medium as a special case of polar-elastic solids, which strain energy density  $W = W(\mathbf{E}, \mathbf{\Gamma})$  is insensitive to rotation by any angle  $\varphi$  about a fixed axis with unit vector  $\mathbf{e}$ . It is possible to prove that the Kelvin medium should in fact be considered as a polar-elastic liquid crystal, because its material symmetry group contains non-orthogonal tensors.

Within non-polar elastic materials C.-C. Wang developed constitutive equations of 14 classes of the simple subfluids being special cases of non-polar liquid crystals, see Wang (1965); Truesdell and Noll (1965); Wang and

Truesdell (1973); Cohen and Wang (1983). The material symmetry group of simple subfluids contained only elements of the dilatation group, that is the group of all unimodular tensors having three fixed linearly independent vectors as their proper vectors. The dilatation group consists of tensors

$$\mathcal{D}_\varkappa = \{\mathbf{P} : \mathbf{P} \in Unim, \mathbf{P} = p_1\mathbf{e}_1 \otimes \mathbf{e}_1 + p_2\mathbf{e}_2 \otimes \mathbf{e}_2 + p_3\mathbf{e}_3 \otimes \mathbf{e}_3, p_1p_2p_3 = \pm 1.\}$$

Here  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are three orthogonal unit vectors called the preferred basis.

In analogy to Wang (1965) we can define polar-elastic subfluids as follows:

**Definition 6.** *The micropolar elastic continuum is called the polar-elastic subfluid at  $x$  and  $\mathbf{B}$  if its material symmetry group  $\mathcal{G}_\varkappa$  contains elements of the dilatation group, that is*

$$\mathcal{G}_\varkappa = \{(\mathbf{P}, \mathbf{R}, \mathbf{L}) : \mathbf{P} \in \mathcal{D}_\varkappa, \mathbf{R} \in Orth, \mathbf{L} \in Lin\}.$$

Discussing application of micropolar elastic continuum for modelling liquid crystals, Eringen (1997) developed constitutive equations based on the strain energy density

$$W = W(\rho, \boldsymbol{\gamma}; \mathbf{j}), \quad (52)$$

where  $\mathbf{j} = \mathbf{Q}\mathbf{J}_\varkappa\mathbf{Q}^T$ ,  $\boldsymbol{\gamma} = \mathbf{Q}\boldsymbol{\Gamma}\mathbf{F}^{-1}$  in our notation. The relative Eulerian wryness measure  $\boldsymbol{\gamma}$  determines relative changes of orientation of the polar-elastic continuum, see Pietraszkiewicz and Eremeyev (2009). From (5) it follows that

$$\boldsymbol{\gamma} = \mathbf{C} - \mathbf{Q}\mathbf{B}\mathbf{F}^{-1}.$$

Hence,  $\boldsymbol{\gamma} = \mathbf{C}$  if and only if  $\mathbf{B} = \mathbf{0}$ . In case of  $\mathbf{B} \neq \mathbf{0}$  the strain energy density  $W$  in (52) corresponds to a special type of polar-elastic continuum, neither a



fluid nor a solid, because  $W$  depends on the relative wryness measure  $\boldsymbol{\gamma}$  which is sensitive to change of orientation of continuum particles with respect to the reference placement. Under change of the reference placement  $\boldsymbol{\gamma}$  transforms according to

$$\boldsymbol{\gamma}_* = \boldsymbol{\gamma} + \mathbf{Q}\mathbf{Z}\mathbf{F}^{-1},$$

where  $\boldsymbol{\gamma}_* = \mathbf{C} - \mathbf{Q}_*\mathbf{B}_*\mathbf{F}_*^{-1}$ .

Using  $\mathbf{j}$  as the structural tensor, Eringen derived various types of constitutive equations. Since in our model  $\mathbf{B}$  is used as the parametric tensor instead of  $\mathbf{j}$ , Eringen's model may be incomparable with ours in sense of the material symmetry group.

Brief discussion stretched above and in Section 5 indicates that polar-elastic liquid crystals and subfluids deserve detailed presentation in a separate paper.

## 10. Some material symmetry groups for polar-elastic solids

Let us discuss simplified forms of  $W$  for some particular cases of anisotropy.

**Definition 7. *Isotropic material.*** *The polar-elastic solid is called isotropic at  $x$  and  $\mathbf{B}$  if there exists a reference placement  $B_x$ , called undistorted, such that the material symmetry group relative to  $B_x$  takes the form*

$$\mathcal{G}_x = \mathcal{S}_x \equiv \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0}) : \mathbf{O} \in Orth\}. \quad (53)$$

This definition means that the strain energy density of the polar-elastic isotropic solid satisfies the relation

$$W(\mathbf{E}, \boldsymbol{\Gamma}; \mathbf{B}) = W[\mathbf{O}\mathbf{E}\mathbf{O}^T, (\det \mathbf{O})\mathbf{O}\boldsymbol{\Gamma}\mathbf{O}^T; (\det \mathbf{O})\mathbf{O}\mathbf{B}\mathbf{O}^T], \quad \forall \mathbf{O} \in Orth.$$

Scalar-valued isotropic functions of a few 2nd-order tensors can be expressed by so-called representation theorems in terms of joint invariants of the tensorial arguments. To construct representation for  $W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B})$  as an isotropic function of joint invariants of  $\mathbf{E}$ ,  $\mathbf{\Gamma}$ , and  $\mathbf{B}$ , we recall first the results of Spencer (1965, 1971) on isotropic scalar-valued functions which are invariant with respect to proper orthogonal tensors  $\mathbf{O} \in Orth^+$ . Then, we apply additional restriction on the representations by requiring also invariance with respect to mirror reflection, i.e. additionally considering  $\mathbf{O} = -\mathbf{I}$ .

Decomposing non-symmetric tensors  $\mathbf{E}$ ,  $\mathbf{\Gamma}$ , and  $\mathbf{B}$  into symmetric and skew parts,

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_S + \mathbf{E}_A, & \mathbf{E}_S &= \frac{1}{2}(\mathbf{E} + \mathbf{E}^T), & \mathbf{E}_A &= \frac{1}{2}(\mathbf{E} - \mathbf{E}^T), \\ \mathbf{\Gamma} &= \mathbf{\Gamma}_S + \mathbf{\Gamma}_A, & \mathbf{\Gamma}_S &= \frac{1}{2}(\mathbf{\Gamma} + \mathbf{\Gamma}^T), & \mathbf{\Gamma}_A &= \frac{1}{2}(\mathbf{\Gamma} - \mathbf{\Gamma}^T), \\ \mathbf{B} &= \mathbf{B}_S + \mathbf{B}_A, & \mathbf{B}_S &= \frac{1}{2}(\mathbf{B} + \mathbf{B}^T), & \mathbf{B}_A &= \frac{1}{2}(\mathbf{B} - \mathbf{B}^T),\end{aligned}$$

we represent  $W$  as scalar-valued function of three symmetric and three skew tensors,

$$W = W(\mathbf{E}_S, \mathbf{E}_A, \mathbf{\Gamma}_S, \mathbf{\Gamma}_A; \mathbf{B}_S, \mathbf{B}_A). \quad (54)$$

Integrity basis for the proper orthogonal group is given by Spencer, see Table 1 in Spencer (1965) or Table II in Spencer (1971). These tables are too lengthy to be presented here. Number of members of integrity basis of  $\mathbf{E}_S, \mathbf{E}_A, \mathbf{\Gamma}_S, \mathbf{\Gamma}_A, \mathbf{B}_S, \mathbf{B}_A$  is much larger than number of components of these tensors. However, there are some polynomial dependencies between elements of the integrity basis of three symmetric and three skew tensors. Hence the number of functionally independent invariants can be reduced, but it is still very large, see Zheng (1994).

For proper orthogonal group there is no difference in transformations of the axial and polar tensors. It is not the case if one considers transformations using the full orthogonal group. Since  $\mathbf{\Gamma}_S, \mathbf{\Gamma}_A, \mathbf{B}_S, \mathbf{B}_A$  are axial tensors, not all invariants listed in Spencer (1965, 1971) are absolute invariants under orthogonal transformations, because some of them change sign under non-proper orthogonal transformations. Following Spencer (1971), we call such invariants relative invariants. Examples of relative invariants are  $\text{tr } \mathbf{\Gamma}_S$ ,  $\text{tr } \mathbf{\Gamma}_S^3$ ,  $\text{tr } \mathbf{E}_S \mathbf{\Gamma}_S$ ,  $\text{tr } \mathbf{E}_S \mathbf{B}_S$ , etc. This gives us the following property of  $W$ :

$$W(\mathbf{E}_S, \mathbf{E}_A, \mathbf{\Gamma}_S, \mathbf{\Gamma}_A; \mathbf{B}_S, \mathbf{B}_A) = W(\mathbf{E}_S, \mathbf{E}_A, -\mathbf{\Gamma}_S, -\mathbf{\Gamma}_A; -\mathbf{B}_S, -\mathbf{B}_A). \quad (55)$$

There are 119 invariants of non-symmetric tensors  $\mathbf{E}, \mathbf{\Gamma}, \mathbf{B}$  presented in Table A.1, see Appendix A. The strain energy density of polar-elastic isotropic solids can be any scalar-valued function of these invariants which satisfy (55).

If we neglect explicit dependence of  $W$  on  $\mathbf{B}$ , or assume that  $\mathbf{B} = \mathbf{0}$ , then  $W = W(\mathbf{E}, \mathbf{\Gamma})$ . Integrity basis of two non-symmetric tensors under the orthogonal group contains 39 members. Following Zheng (1994), Ramezani et al. (2009) listed these invariants for the non-linear polar-elastic solids and proposed corresponding constitutive equations. Let us note, however, that not all 39 elements of this integrity basis are functionally independent. Kafadar and Eringen (1971) constructed the functional basis for two non-symmetric tensors taking into account these functional dependencies. Table A.1 contains the invariants of Ramezani et al. (2009) and of Kafadar and Eringen (1971) as well as additional joint invariants of  $\mathbf{E}, \mathbf{\Gamma}$ , and  $\mathbf{B}$ . According to Kafadar and Eringen (1971), as the isotropic scalar-valued function of two

non-symmetric tensors  $\mathbf{E}$  and  $\mathbf{\Gamma}$ ,  $W$  is expressible in terms of 15 invariants,

$$W = W(I_1, I_2, \dots, I_{15}), \quad (56)$$

where  $I_k$  are given by

$$\begin{aligned} I_1 &= \text{tr } \mathbf{E}, & I_2 &= \text{tr } \mathbf{E}^2, & I_3 &= \text{tr } \mathbf{E}^3, \\ I_4 &= \text{tr } \mathbf{E}\mathbf{E}^T, & I_5 &= \text{tr } \mathbf{E}^2\mathbf{E}^T, & I_6 &= \text{tr } \mathbf{E}^2\mathbf{E}^{T2}, \\ I_7 &= \text{tr } \mathbf{E}\mathbf{\Gamma}, & I_8 &= \text{tr } \mathbf{E}^2\mathbf{\Gamma}, & I_9 &= \text{tr } \mathbf{E}\mathbf{\Gamma}^2, \\ I_{10} &= \text{tr } \mathbf{\Gamma}, & I_{11} &= \text{tr } \mathbf{\Gamma}^2, & I_{12} &= \text{tr } \mathbf{\Gamma}^3, \\ I_{13} &= \text{tr } \mathbf{\Gamma}\mathbf{\Gamma}^T, & I_{14} &= \text{tr } \mathbf{\Gamma}^2\mathbf{\Gamma}^T, & I_{15} &= \text{tr } \mathbf{\Gamma}^2\mathbf{\Gamma}^{T2}. \end{aligned}$$

Taking into account that  $W = W(\mathbf{E}, \mathbf{\Gamma})$  is an even function with respect to  $\mathbf{\Gamma}$ , because in our case the group  $\mathcal{S}_\times$  contains the reflection  $-\mathbb{I}$ ,  $W$  becomes also an even function with respect to some invariants,

$$\begin{aligned} W(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}, I_{11}, I_{12}, I_{13}, I_{14}, I_{15}) & \quad (57) \\ = W(I_1, I_2, I_3, I_4, I_5, I_6, -I_7, -I_8, I_9, -I_{10}, I_{11}, -I_{12}, I_{13}, -I_{14}, I_{15}). \end{aligned}$$

Expanding  $W$  into the Taylor series relative to  $\mathbf{E}$  and  $\mathbf{\Gamma}$ , and keeping up to quadratic terms, we obtain the approximate polynomial representation of (57),

$$\begin{aligned} W = & w_0 + a_1 I_1 + a_2 I_{10} \\ & + b_1 I_1^2 + b_2 I_1 I_{10} + b_3 I_{10}^2 + b_4 I_4 + b_5 I_2 + b_6 I_7 + b_7 I_{11} + b_8 I_{13} \\ & + O(\max(\|\mathbf{E}\|^3, \|\mathbf{\Gamma}\|^3)), \end{aligned}$$

where  $w_0, a_1, a_2, b_1, \dots, b_8$  are material constants. Applying to the latter expression the property (57), we finally obtain the following polynomial representation of  $W$  up to quadratic terms of  $\mathbf{E}$  and  $\mathbf{\Gamma}$ :

$$W = w_0 + a_1 I_1 + b_1 I_1^2 + b_3 I_{10}^2 + b_4 I_4 + b_5 I_2 + b_7 I_{11} + b_8 I_{13}. \quad (58)$$

We may also consider the representation of  $W$  as a sum of two scalar functions each depending on one strain measure,

$$W = W_1(\mathbf{E}) + W_2(\mathbf{\Gamma}). \quad (59)$$

The form (59) was used for example by Ramezani et al. (2009) in order to generalize classical neo-Hookean and Mooney-Rivlin models to polar-elastic solids. As in Section 7, one can use here the representation theorem for isotropic scalar-valued function of one non-symmetric tensor given by Smith and Smith (1971). This leads to the following representation of  $W$ :

$$W = \widetilde{W}_1(I_1, \dots, I_6) + \widetilde{W}_2(I_{10}, \dots, I_{15}), \quad (60)$$

where  $\widetilde{W}_2$  is such that

$$\widetilde{W}_2(I_{10}, I_{11}, I_{12}, I_{13}, I_{14}, I_{15}) = \widetilde{W}_2(-I_{10}, I_{11}, -I_{12}, I_{13}, -I_{14}, I_{15}). \quad (61)$$

Expanding (60) with (61) into the Taylor series and keeping up to quadratic terms in  $\mathbf{E}$  and  $\mathbf{\Gamma}$ ,  $W$  takes the form (59) with

$$W_1 = w_0 + a_1 I_1 + b_1 I_1^2 + b_4 I_4 + b_5 I_2, \quad W_2 = b_3 I_{10}^2 + b_7 I_{11} + b_8 I_{13}.$$

**Definition 8. Hemitropic material.** *The polar-elastic solid is called hemitropic at  $x$  and  $\mathbf{B}$  if there exists a reference placement  $B_x$ , called undistorted, such that the material symmetry group relative to  $B_x$  takes the form*

$$\mathcal{G}_x = \mathcal{S}_x^+ \equiv \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0}) : \mathbf{O} \in Orth^+\}. \quad (62)$$

The strain energy density of the hemitropic polar-elastic solid satisfies the relation

$$W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{B}) = W(\mathbf{OEO}^T, \mathbf{O}\mathbf{\Gamma}\mathbf{O}^T; \mathbf{OBO}^T), \quad \forall \mathbf{O} \in Orth^+. \quad (63)$$

The hemitropic polar-elastic solid is insensitive to change of orientation of the space. In case of reduced strain energy density  $W = W(\mathbf{E}, \mathbf{\Gamma})$  the representation of  $W$  is given by (56), but the property (57) does not hold, in general. Obviously, the polar-elastic isotropic solid is also hemitropic.

Definitions (53) and (63) are somewhat similar to the corresponding definition of the isotropic polar-elastic solid proposed by Eringen and Kafadar (1976), who required

$$W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{J}_x) = W(\mathbf{OEO}^T, \mathbf{O}\mathbf{\Gamma}\mathbf{O}^T; \mathbf{OJ}_x\mathbf{O}^T), \quad \forall \mathbf{O} \in Orth. \quad (64)$$

However, properties (57) or (61) do not follow from (64) and additionally dependence of  $W$  upon  $\mathbf{J}_x$  is disregarded here.

**Definition 9. Orthotropic material.** *The polar-elastic solid is called orthotropic at  $x$  and  $\mathbf{B}$  if the material symmetry group for some reference placement  $B_x$  takes the form*

$$\begin{aligned} \mathcal{G}_x &= \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0})\} : \\ &\mathbf{O} = \{\mathbf{I}, -\mathbf{I}, 2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{I}, 2\mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{I}, 2\mathbf{e}_3 \otimes \mathbf{e}_3 - \mathbf{I}\}, \end{aligned} \quad (65)$$

where  $\mathbf{O}$  are orthogonal tensors performing mirror reflections and rotations of  $180^\circ$  about three orthonormal vectors  $\mathbf{e}_k$ .

Obviously, the polar-elastic isotropic solid is also orthotropic. Thus, all invariants in Tables A.1 can be used for representation of the strain energy

density of the polar-elastic orthotropic solid. There are 60 additional absolute and relative invariants responsible for orthotropic properties of polar-elastic solids, so that the total number of invariants becomes 179.

**Definition 10. Cubic symmetry.** *The polar-elastic solid is called cubic-symmetric at  $x$  and  $\mathbf{B}$  if the material symmetry group for some reference placement  $B_x$  takes the form*

$$\mathcal{G}_x = \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0})\} : \quad \mathbf{O} = \{\mathbf{I}, -\mathbf{I}, \mathbf{e}_1 \otimes \mathbf{e}_1 \mp \mathbf{e}_1 \times \mathbf{I}, \mathbf{e}_2 \otimes \mathbf{e}_2 \mp \mathbf{e}_2 \times \mathbf{I}, \mathbf{e}_3 \otimes \mathbf{e}_3 \mp \mathbf{e}_3 \times \mathbf{I}\}, \quad (66)$$

where  $\mathbf{O}$  are orthogonal tensors performing mirror reflections and rotations of  $90^\circ$  about three orthonormal vectors  $\mathbf{e}_k$ .

**Definition 11. Transversely isotropic material.** *The polar-elastic solid is called transversely isotropic at  $x$  and  $\mathbf{B}$  with respect to a direction described by  $\mathbf{e}$  if the material symmetry group for some reference placement  $B_x$  takes the form*

$$\mathcal{G}_x = \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0})\} : \quad \mathbf{O} = \{\mathbf{I}, -\mathbf{I}, \mathbf{O}(\varphi\mathbf{e}), \quad \forall \varphi\}, \quad (67)$$

where  $\mathbf{O}(\varphi\mathbf{e}) = (\mathbf{I} - \mathbf{e} \otimes \mathbf{e}) \cos \varphi + \mathbf{e} \otimes \mathbf{e} + \mathbf{e} \times \mathbf{I} \sin \varphi$  is the rotation tensor with the rotation angle  $\varphi$  about the unit vector  $\mathbf{e}$ .

There are 167 absolute and relative invariants for polar-elastic transversely isotropic solids.

## 11. Physically linear polar-elastic solids

Let us consider the polar-elastic strain energy density as a quadratic function of  $\mathbf{E}$  and  $\mathbf{\Gamma}$ ,

$$W = \frac{1}{2} \mathbf{E} : \mathbf{C} : \mathbf{E} + \mathbf{E} : \mathbf{B} : \mathbf{\Gamma} + \frac{1}{2} \mathbf{\Gamma} : \mathbf{D} : \mathbf{\Gamma}, \quad (68)$$

where  $\mathbf{C}$ ,  $\mathbf{B}$ , and  $\mathbf{D}$  are 4th-order tensors of elastic moduli of the polar-elastic solid. Components of tensors  $\mathbf{C}$  and  $\mathbf{D}$  have symmetry properties

$$C_{ijmn} = C_{mnij}, \quad D_{ijmn} = D_{mnij}.$$

With (68) corresponding referential stress measures  $\mathbf{S}$  and  $\mathbf{K}$  follow from (8) to be

$$\mathbf{S} = \mathbf{C} : \mathbf{E} + \mathbf{B} : \mathbf{\Gamma}, \quad \mathbf{K} = \mathbf{E} : \mathbf{B} + \mathbf{D} : \mathbf{\Gamma}.$$

The model based on (68) can be called the physically linear polar-elastic solid. In this case tensors  $\mathbf{C}$ ,  $\mathbf{B}$ , and  $\mathbf{D}$  coincide with the tangent stiffness tensors introduced by Ramezani et al. (2009). Since  $\mathbf{E}$  and  $\mathbf{\Gamma}$  are non-symmetric tensors,  $W$  contains  $2 \cdot 45 + 81 = 171$  independent material constants, in general. In what follows we find restrictions for  $\mathbf{C}$ ,  $\mathbf{B}$ , and  $\mathbf{D}$  corresponding to some particular material symmetry groups.

**1. Mirror reflection.** Let  $\mathcal{S}_\varkappa$  contain two elements,  $\mathcal{S}_\varkappa = \{\mathbb{I}, -\mathbb{I}\}$ . Substituting  $-\mathbb{I}$  into (51) we obtain

$$\frac{1}{2} \mathbf{E} : \mathbf{C} : \mathbf{E} + \mathbf{E} : \mathbf{B} : \mathbf{\Gamma} + \frac{1}{2} \mathbf{\Gamma} : \mathbf{D} : \mathbf{\Gamma} = \frac{1}{2} \mathbf{E} : \mathbf{C} : \mathbf{E} - \mathbf{E} : \mathbf{B} : \mathbf{\Gamma} + \frac{1}{2} \mathbf{\Gamma} : \mathbf{D} : \mathbf{\Gamma},$$

from which it follows that either  $\mathbf{B} = \mathbf{O}$ , where  $\mathbf{O}$  is the 4th-order zeroth tensor, or as has been shown above (Case 1 in Section 5),  $W$  is an even



function of  $\mathbf{E}$  and  $\mathbf{\Gamma}$ . In this case one has 90 independent scalar elastic moduli.

In what follows we always assume that  $\mathcal{S}_z$  contains  $-\mathbb{I}$  with only one exception for the hemitropic solid. Hence,  $W$ ,  $\mathbf{S}$ , and  $\mathbf{K}$  take the reduced form

$$W = \frac{1}{2} \mathbf{E} : \mathbf{C} : \mathbf{E} + \frac{1}{2} \mathbf{\Gamma} : \mathbf{D} : \mathbf{\Gamma}, \quad \mathbf{S} = \mathbf{C} : \mathbf{E}, \quad \mathbf{K} = \mathbf{D} : \mathbf{\Gamma}. \quad (69)$$

In the tensor base comprised of directors  $\mathbf{h}_a$  the tensors  $\mathbf{C}$  and  $\mathbf{D}$  can be expressed as symmetric  $9 \times 9$  matrices. For example, the matrix of  $\mathbf{C}$  is

$$[\mathbf{C}] = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\ & & C_{3333} & C_{3323} & C_{3331} & C_{3312} & C_{3332} & C_{3313} & C_{3321} \\ & & & C_{2323} & C_{2331} & C_{2312} & C_{2332} & C_{2313} & C_{2321} \\ & & & & C_{3131} & C_{3112} & C_{3132} & C_{3113} & C_{3121} \\ & & & & & C_{1212} & C_{1232} & C_{1213} & C_{1221} \\ & & & & & & C_{3232} & C_{3213} & C_{3221} \\ & & & & & & & C_{1313} & C_{1321} \\ & & & & & & & & C_{2121} \end{pmatrix},$$

Since  $[\mathbf{D}]$  has the same structure as  $[\mathbf{C}]$ , in what follows we restrict ourselves to representation of  $[\mathbf{C}]$ .

**2. Monoclinic symmetry.** The corresponding material symmetry group  $\mathcal{S}_z$  consists of mirror reflection and rotations of  $180^\circ$  about unit vector  $\mathbf{e}$ :  $\mathbf{O} = \{\mathbf{I}, -\mathbf{I}, 2\mathbf{e} \otimes \mathbf{e} - \mathbf{I}\}$ . Using  $\mathbf{e}$  as coordinate vector of the 3rd axis, we obtain that matrices  $[\mathbf{C}]$  and  $[\mathbf{D}]$  should have some zero elements, so that  $[\mathbf{C}]$

takes the form

$$[\mathbf{C}] = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} & 0 & 0 & C_{1121} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} & 0 & 0 & C_{2221} \\ & & C_{3333} & 0 & 0 & C_{3312} & 0 & 0 & C_{3321} \\ & & & C_{2323} & C_{2331} & 0 & C_{2332} & C_{2313} & 0 \\ & & & & C_{3131} & 0 & C_{3132} & C_{3113} & 0 \\ & & & & & C_{1212} & 0 & 0 & C_{1221} \\ & & SYM & & & & C_{3232} & C_{3213} & 0 \\ & & & & & & & C_{1313} & 0 \\ & & & & & & & & C_{2121} \end{pmatrix}.$$

The polar-elastic physically linear monoclinic solid is described by 50 independent elastic moduli.

**3. Orthotropic symmetry.** In this case we have 30 independent elastic moduli. The corresponding matrix  $[\mathbf{C}]$  is given by

$$[\mathbf{C}] = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & C_{2323} & 0 & 0 & C_{2332} & 0 & 0 \\ & & & & C_{3131} & 0 & 0 & C_{3113} & 0 \\ & & & & & C_{1212} & 0 & 0 & C_{1221} \\ & & SYM & & & & C_{3232} & 0 & 0 \\ & & & & & & & C_{1313} & 0 \\ & & & & & & & & C_{2121} \end{pmatrix}.$$

**4. Rotations by  $90^\circ$ .** Additionally to the previous case, let us require invariance of  $W$  under rotations of  $90^\circ$  about the unit vector  $\mathbf{e}_3$ :  $\mathbf{O} = \{\mathbf{e}_3 \otimes$

$\mathbf{e}_3 \mp \mathbf{e}_3 \times \mathbf{I}$ . As a result, the solid is described by 18 independent elastic moduli and  $[\mathbf{C}]$  is given by

$$[\mathbf{C}] = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ & C_{1111} & C_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & C_{2323} & 0 & 0 & C_{2332} & 0 & 0 \\ & & & & C_{3131} & 0 & 0 & C_{2332} & 0 \\ & & & & & C_{1212} & 0 & 0 & C_{1221} \\ & & SYM & & & & C_{3131} & 0 & 0 \\ & & & & & & & C_{2323} & 0 \\ & & & & & & & & C_{1212} \end{pmatrix}.$$

**5. Cubic symmetry.** The material symmetry group  $\mathcal{G}_\times$  related to cubic symmetry additionally contains rotations of  $90^\circ$  about all orthogonal axes with unit vectors  $\mathbf{e}_k$ ,  $k = 1, 2, 3$ . Hence, the matrix  $[\mathbf{C}]$  becomes

$$[\mathbf{C}] = \begin{pmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & C_{1212} & 0 & 0 & C_{1221} & 0 & 0 \\ & & & & C_{1212} & 0 & 0 & C_{1221} & 0 \\ & & & & & C_{1212} & 0 & 0 & C_{1221} \\ & & SYM & & & & C_{1212} & 0 & 0 \\ & & & & & & & C_{1212} & 0 \\ & & & & & & & & C_{1212} \end{pmatrix}.$$

For the physically linear polar-elastic solid with cubic symmetry we have eight independent elastic moduli.

**6. Isotropy.** In this case one can use representation (56) with (57). Taking into account only quadratic terms in  $\mathbf{E}$  and  $\mathbf{\Gamma}$  we find that

$$2W = \alpha_1 I_1^2 + \alpha_2 I_2 + \alpha_3 I_4 + \beta_1 I_{10}^2 + \beta_2 I_{11} + \beta_3 I_{13}, \quad (70)$$

where  $\alpha_a, \beta_a, a = 1, 2, 3$  are independent elastic moduli. The corresponding tensors  $\mathbf{C}$  and  $\mathbf{D}$  are

$$\mathbf{C} = \alpha_1 \mathbf{I} \otimes \mathbf{I} + \alpha_2 \mathbf{h}_a \otimes \mathbf{I} \otimes \mathbf{h}_a + \alpha_3 \mathbf{h}_a \otimes \mathbf{h}_b \otimes \mathbf{h}_a \otimes \mathbf{h}_b, \quad (71)$$

$$\mathbf{D} = \beta_1 \mathbf{I} \otimes \mathbf{I} + \beta_2 \mathbf{h}_a \otimes \mathbf{I} \otimes \mathbf{h}_a + \beta_3 \mathbf{h}_a \otimes \mathbf{h}_b \otimes \mathbf{h}_a \otimes \mathbf{h}_b, \quad (72)$$

where  $\mathbf{h}_a, a = 1, 2, 3$  is now any Cartesian base.

The strain energy density of the physically linear polar-elastic isotropic solid contains only six scalar elastic moduli. This constitutive relation corresponds to the linear isotropic elastic Cosserat continuum widely used in the literature, see for example Nowacki (1986); Eringen (1999). For example, in terms of material constants used by Eringen (1999), formulas 5.3.1, our constants  $\alpha_a, \beta_a$  are expressed as follows:  $\alpha_1 = \lambda, \quad \alpha_2 = \mu, \quad \alpha_3 = \mu + \kappa, \quad \beta_1 = \alpha, \quad \beta_2 = \beta, \quad \beta_3 = \gamma.$

Within linear micropolar elasticity the explicit structure of tensors  $\mathbf{C}$  and  $\mathbf{D}$  was presented by Zheng and Spencer (1993) for 14 symmetry groups.

**7. Hemitropy.** For the polar-elastic hemitropic solid the material symmetry group contains only rotation tensors, see (62). Hence  $\mathbf{B} \neq \mathbf{O}$ , in general. Tensors  $\mathbf{C}$  and  $\mathbf{D}$  take here the form (71) while  $\mathbf{B}$  is given by

$$\mathbf{B} = \gamma_1 \mathbf{I} \otimes \mathbf{I} + \gamma_2 \mathbf{h}_a \otimes \mathbf{I} \otimes \mathbf{h}_a + \gamma_3 \mathbf{h}_a \otimes \mathbf{h}_b \otimes \mathbf{h}_a \otimes \mathbf{h}_b,$$

where  $\gamma_a, a = 1, 2, 3$ , are additional independent scalar elastic moduli. As a result, for the polar-elastic physically linear hemitropic solid the strain energy

density contains nine independent scalar elastic moduli. In linear micropolar elasticity this representation is given for example in Dyszlewicz (2004).

## 12. Physically linear polar-elastic solids which elasticity tensors depend on the microstructure curvature tensor

Let us now consider the polar-elastic strain energy density as a quadratic function (68) of  $\mathbf{E}$  and  $\mathbf{\Gamma}$ ,

$$W = \frac{1}{2} \mathbf{E} : \mathbf{C}(\mathbf{B}) : \mathbf{E} + \mathbf{E} : \mathbf{B}(\mathbf{B}) : \mathbf{\Gamma} + \frac{1}{2} \mathbf{\Gamma} : \mathbf{D}(\mathbf{B}) : \mathbf{\Gamma}. \quad (73)$$

But now  $\mathbf{C}$ ,  $\mathbf{B}$ , and  $\mathbf{D}$  are 4th-order tensors of elastic moduli assumed to depend on  $\mathbf{B}$ . The tensor  $\mathbf{B}$  plays here the role of structural tensor used in representation of anisotropic tensor functions, see for example Boehler (1987) and Zheng (1994). Dependence of elasticity tensors on  $\mathbf{B}$  changes significantly representation of (73) and the number of independent elastic moduli.

In what follows, for simplicity, we restrict ourselves to polar-elastic hemitropic and isotropic solids. Using Table A.1, we obtain the following 58 quadratic invariants of  $\mathbf{E}$  and  $\mathbf{\Gamma}$ :

$$\begin{aligned} J_1 &= \text{tr}^2 \mathbf{E}_S, & J_2 &= \text{tr} \mathbf{E}_S^2, & J_3 &= \text{tr} \mathbf{E}_A^2, \\ J_4 &= \text{tr}^2 \mathbf{\Gamma}_S, & J_5 &= \text{tr} \mathbf{\Gamma}_S^2, & J_6 &= \text{tr} \mathbf{\Gamma}_A^2, \\ \underline{J_7} &= \text{tr} \mathbf{E}_S \mathbf{\Gamma}_S, & & & & \\ J_8 &= \text{tr}^2 \mathbf{E}_S \mathbf{B}_S, & \underline{J_9} &= \text{tr} \mathbf{E}_S^2 \mathbf{B}_S, & & \\ J_{10} &= \text{tr}^2 \mathbf{E}_S \mathbf{B}_S^2, & J_{11} &= \text{tr} \mathbf{E}_S^2 \mathbf{B}_S^2, & & \\ J_{12} &= \text{tr}^2 \mathbf{\Gamma}_S \mathbf{B}_S, & \underline{J_{13}} &= \text{tr} \mathbf{\Gamma}_S^2 \mathbf{B}_S, & & \end{aligned} \quad (74)$$

$$\begin{aligned}
J_{14} &= \text{tr}^2 \boldsymbol{\Gamma}_S \mathbf{B}_S^2, & J_{15} &= \text{tr} \boldsymbol{\Gamma}_S^2 \mathbf{B}_S^2, & J_{16} &= \text{tr} \mathbf{E}_S \boldsymbol{\Gamma}_S \mathbf{B}_S, \\
\underline{J_{17}} &= \text{tr} \mathbf{E}_A \boldsymbol{\Gamma}_A, & J_{18} &= \text{tr}^2 \mathbf{E}_A \mathbf{B}_A, \\
J_{19} &= \text{tr}^2 \boldsymbol{\Gamma}_A \mathbf{B}_A, & J_{20} &= \text{tr} \mathbf{E}_A \boldsymbol{\Gamma}_A \mathbf{B}_A, \\
J_{21} &= \text{tr}^2 \mathbf{E}_S \mathbf{B}_A^2, & J_{22} &= \text{tr} \mathbf{E}_S^2 \mathbf{B}_A^2, \\
J_{23} &= \text{tr}^2 \boldsymbol{\Gamma}_S \mathbf{B}_A^2, & J_{24} &= \text{tr} \boldsymbol{\Gamma}_S^2 \mathbf{B}_A^2, \\
\underline{J_{25}} &= \text{tr} \mathbf{B}_S \boldsymbol{\Gamma}_A^2, & J_{26} &= \text{tr} \mathbf{B}_S^2 \boldsymbol{\Gamma}_A^2, \\
\underline{J_{27}} &= \text{tr} \mathbf{B}_S \mathbf{E}_A^2, & J_{28} &= \text{tr} \mathbf{B}_S^2 \mathbf{E}_A^2, \\
J_{29} &= \text{tr} \mathbf{E}_S \boldsymbol{\Gamma}_S \mathbf{B}_A, & \underline{J_{30}} &= \text{tr} \mathbf{E}_S \mathbf{B}_S \mathbf{E}_A, \\
J_{31} &= \text{tr} \mathbf{E}_S \mathbf{B}_S \boldsymbol{\Gamma}_A, & \underline{J_{32}} &= \text{tr} \mathbf{E}_S \mathbf{B}_S^2 \boldsymbol{\Gamma}_A, \\
J_{33} &= \text{tr}^2 \mathbf{E}_S \mathbf{B}_S \mathbf{B}_A, & J_{34} &= \text{tr} \mathbf{E}_S^2 \mathbf{B}_S \mathbf{B}_A, \\
J_{35} &= \text{tr}^2 \mathbf{E}_S \mathbf{B}_S^2 \mathbf{B}_A, & J_{36} &= \text{tr}^2 \mathbf{E}_S \mathbf{B}_A^2 \mathbf{B}_S \mathbf{B}_A, \\
J_{37} &= \text{tr} \boldsymbol{\Gamma}_S \mathbf{B}_S \mathbf{E}_A, & J_{38} &= \text{tr} \boldsymbol{\Gamma}_S \mathbf{B}_S \boldsymbol{\Gamma}_A, & J_{39} &= \text{tr} \boldsymbol{\Gamma}_S \mathbf{B}_S^2 \boldsymbol{\Gamma}_A, \\
J_{40} &= \text{tr}^2 \boldsymbol{\Gamma}_S \mathbf{B}_S \mathbf{B}_A, & J_{41} &= \text{tr} \boldsymbol{\Gamma}_S^2 \mathbf{B}_S \mathbf{B}_A, \\
J_{42} &= \text{tr}^2 \boldsymbol{\Gamma}_S \mathbf{B}_S^2 \mathbf{B}_A, & J_{43} &= \text{tr}^2 \boldsymbol{\Gamma}_S \mathbf{B}_A^2 \mathbf{B}_S \mathbf{B}_A, \\
\underline{J_{44}} &= \text{tr} \mathbf{E}_S \mathbf{E}_A \mathbf{B}_A, & J_{45} &= \text{tr} \mathbf{E}_S \mathbf{E}_A \mathbf{B}_A^2, \\
J_{46} &= \text{tr} \mathbf{E}_S \boldsymbol{\Gamma}_A \mathbf{B}_A, & \underline{J_{47}} &= \text{tr} \mathbf{E}_S \boldsymbol{\Gamma}_A \mathbf{B}_A^2, \\
J_{48} &= \text{tr} \boldsymbol{\Gamma}_S \mathbf{E}_A \mathbf{B}_A, & J_{49} &= \text{tr} \boldsymbol{\Gamma}_S \mathbf{E}_A \mathbf{B}_A^2, \\
J_{50} &= \text{tr} \boldsymbol{\Gamma}_S \boldsymbol{\Gamma}_A \mathbf{B}_A, & J_{51} &= \text{tr} \boldsymbol{\Gamma}_S \boldsymbol{\Gamma}_A \mathbf{B}_A^2, & J_{52} &= \text{tr} \mathbf{B}_S \mathbf{E}_A \boldsymbol{\Gamma}_A, \\
J_{53} &= \text{tr}^2 \mathbf{B}_S \mathbf{E}_A \mathbf{B}_A, & J_{54} &= \text{tr} \mathbf{B}_S \mathbf{E}_A^2 \mathbf{B}_A, & J_{55} &= \text{tr}^2 \mathbf{B}_S \mathbf{E}_A \mathbf{B}_A^2, \\
J_{56} &= \text{tr}^2 \mathbf{B}_S \boldsymbol{\Gamma}_A \mathbf{B}_A, & J_{57} &= \text{tr} \mathbf{B}_S \boldsymbol{\Gamma}_A^2 \mathbf{B}_A, & J_{58} &= \text{tr}^2 \mathbf{B}_S \boldsymbol{\Gamma}_A \mathbf{B}_A^2.
\end{aligned}$$

The strain energy density of linear polar-elastic hemitropic solid is now the sum of 58 terms,

$$W = \sum_{i=1}^{58} c_i J_i, \quad (75)$$

where  $c_i$  are functions of invariants of  $\mathbf{B}$ , in general.

Since underlined terms in (74) are relative invariants, which change signs under change of orientation of the space, for the polar-elastic isotropic solid the strain energy density takes the form (75), but now with  $c_7 = c_9 = c_{13} = c_{25} = c_{27} = c_{30} = c_{32} = c_{44} = c_{47} = 0$ .

One can derive in the same manner quadratic strain energy densities of polar-elastic orthotropic and transversely isotropic solids. Let us however note that when one takes into account explicit dependence of elasticity tensors on  $\mathbf{B}$  one obtains a huge number of material parameters even when one assumes that  $c_i$  are constant elastic moduli.

### 13. Conclusions

We have defined anew the material symmetry group  $\mathcal{G}_\varkappa$  of the non-linear polar continuum. The group generalizes the one proposed by Eringen and Kafadar (1976) by introducing the undeformed microstructure curvature tensor  $\mathbf{B}$  instead of the microinertia tensor  $\mathbf{J}_\varkappa$  as well as by taking into account that  $\mathbf{B}$  and the wryness tensor  $\mathbf{\Gamma}$  are axial tensors which change signs under inversion transformation of 3D space. Our group  $\mathcal{G}_\varkappa$  consists of an ordered triple of tensors which make the strain energy density invariant under change of reference placement. In terms of members of  $\mathcal{G}_\varkappa$  polar-elastic fluids, solids, liquid crystals, and subfluids have conveniently been defined. Reduced forms of constitutive equations for non-linear and physically linear polar-elastic solids are given for several particular material symmetry groups.

In many cases discussed here even the reduced constitutive equations of polar-elastic solids still involve a large number of material constants, which should be experimentally or theoretically identified. Difficulties in identifica-

tions of material constants pose a serious challenge for wider application of polar-elastic solids in science and technology.

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## Appendix A. Invariants in strain energy density

Using the representations of Zheng (1994), for isotropic polar-elastic solids in Table A.1 we present the list of 119 irreducible invariants of  $\mathbf{E}$ ,  $\mathbf{\Gamma}$ , and  $\mathbf{B}$  expressed in terms of their symmetric and skew-symmetric parts.

Table A.1: Invariants in  $W$

Agencies	Invariants			
$\mathbf{E}_S$	$\text{tr } \mathbf{E}_S$	$\text{tr } \mathbf{E}_S^2$	$\text{tr } \mathbf{E}_S^3$	
$\mathbf{E}_A$	$\text{tr } \mathbf{E}_A^2$			
$\mathbf{E}_S, \mathbf{E}_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A^2$	$\text{tr } \mathbf{E}_S^2 \mathbf{E}_A^2$	$\text{tr } \mathbf{E}_S^2 \mathbf{E}_A^2 \mathbf{E}_S \mathbf{E}_A$	
$\mathbf{\Gamma}_S$	$\text{tr } \mathbf{\Gamma}_S$	$\text{tr } \mathbf{\Gamma}_S^2$	$\text{tr } \mathbf{\Gamma}_S^3$	
$\mathbf{\Gamma}_A$	$\text{tr } \mathbf{\Gamma}_A^2$			
$\mathbf{\Gamma}_S, \mathbf{\Gamma}_A$	$\text{tr } \mathbf{\Gamma}_S \mathbf{\Gamma}_A^2$	$\text{tr } \mathbf{\Gamma}_S^2 \mathbf{\Gamma}_A^2$	$\text{tr } \mathbf{\Gamma}_S^2 \mathbf{\Gamma}_A^2 \mathbf{\Gamma}_S \mathbf{\Gamma}_A$	
$\mathbf{B}_S$	$\text{tr } \mathbf{B}_S$	$\text{tr } \mathbf{B}_S^2$	$\text{tr } \mathbf{B}_S^3$	
$\mathbf{B}_A$	$\text{tr } \mathbf{B}_A^2$			
$\mathbf{B}_S, \mathbf{B}_A$	$\text{tr } \mathbf{B}_S \mathbf{B}_A^2$	$\text{tr } \mathbf{B}_S^2 \mathbf{B}_A^2$	$\text{tr } \mathbf{B}_S^2 \mathbf{B}_A^2 \mathbf{B}_S \mathbf{B}_A$	
$\mathbf{E}_S, \mathbf{\Gamma}_S$	$\text{tr } \mathbf{E}_S \mathbf{\Gamma}_S$	$\text{tr } \mathbf{E}_S^2 \mathbf{\Gamma}_S$	$\text{tr } \mathbf{E}_S \mathbf{\Gamma}_S^2$	$\text{tr } \mathbf{E}_S^2 \mathbf{\Gamma}_S^2$
$\mathbf{E}_S, \mathbf{B}_S$	$\text{tr } \mathbf{E}_S \mathbf{B}_S$	$\text{tr } \mathbf{E}_S^2 \mathbf{B}_S$	$\text{tr } \mathbf{E}_S \mathbf{B}_S^2$	$\text{tr } \mathbf{E}_S^2 \mathbf{B}_S^2$
$\mathbf{\Gamma}_S, \mathbf{B}_S$	$\text{tr } \mathbf{\Gamma}_S \mathbf{B}_S$	$\text{tr } \mathbf{\Gamma}_S^2 \mathbf{B}_S$	$\text{tr } \mathbf{\Gamma}_S \mathbf{B}_S^2$	$\text{tr } \mathbf{\Gamma}_S^2 \mathbf{B}_S^2$
$\mathbf{E}_S, \mathbf{\Gamma}_S, \mathbf{B}_S$	$\text{tr } \mathbf{E}_S \mathbf{\Gamma}_S \mathbf{B}_S$			
$\mathbf{E}_A, \mathbf{\Gamma}_A$	$\text{tr } \mathbf{E}_A \mathbf{\Gamma}_A$			
$\mathbf{E}_A, \mathbf{B}_A$	$\text{tr } \mathbf{E}_A \mathbf{B}_A$			
$\mathbf{\Gamma}_A, \mathbf{B}_A$	$\text{tr } \mathbf{\Gamma}_A \mathbf{B}_A$			
$\mathbf{E}_A, \mathbf{\Gamma}_A, \mathbf{B}_A$	$\text{tr } \mathbf{E}_A \mathbf{\Gamma}_A \mathbf{B}_A$			
$\mathbf{E}_S, \mathbf{\Gamma}_A$	$\text{tr } \mathbf{E}_S \mathbf{\Gamma}_A^2$	$\text{tr } \mathbf{E}_S^2 \mathbf{\Gamma}_A^2$	$\text{tr } \mathbf{E}_S^2 \mathbf{\Gamma}_A^2 \mathbf{E}_S \mathbf{\Gamma}_A$	

Table A.1: Invariants in  $W$ 

$\mathbf{E}_S, \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_A^2$	$\text{tr } \mathbf{E}_S^2 \mathbf{B}_A^2$	$\text{tr } \mathbf{E}_S^2 \mathbf{B}_A^2 \mathbf{E}_S \mathbf{B}_A$	
$\Gamma_S, \mathbf{E}_A$	$\text{tr } \Gamma_S \mathbf{E}_A^2$	$\text{tr } \Gamma_S^2 \mathbf{E}_A^2$	$\text{tr } \Gamma_S^2 \mathbf{E}_A^2 \Gamma_S \mathbf{E}_A$	
$\Gamma_S, \mathbf{B}_A$	$\text{tr } \Gamma_S \mathbf{B}_A^2$	$\text{tr } \Gamma_S^2 \mathbf{B}_A^2$	$\text{tr } \Gamma_S^2 \mathbf{B}_A^2 \Gamma_S \mathbf{B}_A$	
$\mathbf{B}_S, \Gamma_A$	$\text{tr } \mathbf{B}_S \Gamma_A^2$	$\text{tr } \mathbf{B}_S^2 \Gamma_A^2$	$\text{tr } \mathbf{E}_S^2 \Gamma_A^2 \mathbf{B}_S \Gamma_A$	
$\mathbf{B}_S, \mathbf{E}_A$	$\text{tr } \mathbf{B}_S \mathbf{E}_A^2$	$\text{tr } \mathbf{B}_S^2 \mathbf{E}_A^2$	$\text{tr } \mathbf{B}_S^2 \mathbf{E}_A^2 \mathbf{B}_S \mathbf{E}_A$	
$\mathbf{E}_S, \Gamma_S, \mathbf{E}_A$	$\text{tr } \mathbf{E}_S \Gamma_S \mathbf{E}_A$	$\text{tr } \mathbf{E}_S^2 \Gamma_S \mathbf{E}_A$	$\text{tr } \mathbf{E}_S \Gamma_S^2 \mathbf{E}_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A^2 \Gamma_S \mathbf{E}_A$
$\mathbf{E}_S, \Gamma_S, \Gamma_A$	$\text{tr } \mathbf{E}_S \Gamma_S \Gamma_A$	$\text{tr } \mathbf{E}_S^2 \Gamma_S \Gamma_A$	$\text{tr } \mathbf{E}_S \Gamma_S^2 \Gamma_A$	$\text{tr } \mathbf{E}_S \Gamma_A^2 \Gamma_S \Gamma_A$
$\mathbf{E}_S, \Gamma_S, \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \Gamma_S \mathbf{B}_A$	$\text{tr } \mathbf{E}_S^2 \Gamma_S \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \Gamma_S^2 \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_A^2 \Gamma_S \mathbf{B}_A$
$\mathbf{E}_S, \mathbf{B}_S, \mathbf{E}_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_S \mathbf{E}_A$	$\text{tr } \mathbf{E}_S^2 \mathbf{B}_S \mathbf{E}_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_S^2 \mathbf{E}_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A^2 \mathbf{B}_S \mathbf{E}_A$
$\mathbf{E}_S, \mathbf{B}_S, \Gamma_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_S \Gamma_A$	$\text{tr } \mathbf{E}_S^2 \mathbf{B}_S \Gamma_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_S^2 \Gamma_A$	$\text{tr } \mathbf{E}_S \Gamma_A^2 \mathbf{B}_S \Gamma_A$
$\mathbf{E}_S, \mathbf{B}_S, \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_S \mathbf{B}_A$	$\text{tr } \mathbf{E}_S^2 \mathbf{B}_S \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_S^2 \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \mathbf{B}_A^2 \mathbf{B}_S \mathbf{B}_A$
$\Gamma_S, \mathbf{B}_S, \mathbf{E}_A$	$\text{tr } \Gamma_S \mathbf{B}_S \mathbf{E}_A$	$\text{tr } \Gamma_S^2 \mathbf{B}_S \mathbf{E}_A$	$\text{tr } \Gamma_S \mathbf{B}_S^2 \mathbf{E}_A$	$\text{tr } \Gamma_S \mathbf{E}_A^2 \mathbf{B}_S \mathbf{E}_A$
$\Gamma_S, \mathbf{B}_S, \Gamma_A$	$\text{tr } \Gamma_S \mathbf{B}_S \Gamma_A$	$\text{tr } \Gamma_S^2 \mathbf{B}_S \Gamma_A$	$\text{tr } \Gamma_S \mathbf{B}_S^2 \Gamma_A$	$\text{tr } \Gamma_S \Gamma_A^2 \mathbf{B}_S \Gamma_A$
$\Gamma_S, \mathbf{B}_S, \mathbf{B}_A$	$\text{tr } \Gamma_S \mathbf{B}_S \mathbf{B}_A$	$\text{tr } \Gamma_S^2 \mathbf{B}_S \mathbf{B}_A$	$\text{tr } \Gamma_S \mathbf{B}_S^2 \mathbf{B}_A$	$\text{tr } \Gamma_S \mathbf{B}_A^2 \mathbf{B}_S \mathbf{B}_A$
$\mathbf{E}_S, \mathbf{E}_A, \Gamma_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A \Gamma_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A^2 \Gamma_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A \Gamma_A^2$	
$\mathbf{E}_S, \mathbf{E}_A, \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A^2 \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \mathbf{E}_A \mathbf{B}_A^2$	
$\mathbf{E}_S, \Gamma_A, \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \Gamma_A \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \Gamma_A^2 \mathbf{B}_A$	$\text{tr } \mathbf{E}_S \Gamma_A \mathbf{B}_A^2$	
$\Gamma_S, \mathbf{E}_A, \Gamma_A$	$\text{tr } \Gamma_S \mathbf{E}_A \Gamma_A$	$\text{tr } \Gamma_S \mathbf{E}_A^2 \Gamma_A$	$\text{tr } \Gamma_S \mathbf{E}_A \Gamma_A^2$	
$\Gamma_S, \mathbf{E}_A, \mathbf{B}_A$	$\text{tr } \Gamma_S \mathbf{E}_A \mathbf{B}_A$	$\text{tr } \Gamma_S \mathbf{E}_A^2 \mathbf{B}_A$	$\text{tr } \Gamma_S \mathbf{E}_A \mathbf{B}_A^2$	
$\Gamma_S, \Gamma_A, \mathbf{B}_A$	$\text{tr } \Gamma_S \Gamma_A \mathbf{B}_A$	$\text{tr } \Gamma_S \Gamma_A^2 \mathbf{B}_A$	$\text{tr } \Gamma_S \Gamma_A \mathbf{B}_A^2$	
$\mathbf{B}_S, \mathbf{E}_A, \Gamma_A$	$\text{tr } \mathbf{B}_S \mathbf{E}_A \Gamma_A$	$\text{tr } \mathbf{B}_S \mathbf{E}_A^2 \Gamma_A$	$\text{tr } \mathbf{B}_S \mathbf{E}_A \Gamma_A^2$	
$\mathbf{B}_S, \mathbf{E}_A, \mathbf{B}_A$	$\text{tr } \mathbf{B}_S \mathbf{E}_A \mathbf{B}_A$	$\text{tr } \mathbf{B}_S \mathbf{E}_A^2 \mathbf{B}_A$	$\text{tr } \mathbf{B}_S \mathbf{E}_A \mathbf{B}_A^2$	
$\mathbf{B}_S, \Gamma_A, \mathbf{B}_A$	$\text{tr } \mathbf{B}_S \Gamma_A \mathbf{B}_A$	$\text{tr } \mathbf{B}_S \Gamma_A^2 \mathbf{B}_A$	$\text{tr } \mathbf{B}_S \Gamma_A \mathbf{B}_A^2$	