Material Symmetry Group and Consistently Reduced Constitutive Equations of the Elastic Cosserat Continuum

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Abstract We discuss the material symmetry group of the polar-elastic continuum and related consistently simplified constitutive equations. Following [3] we extent the definition of the group proposed by Eringen and Kafadar [6] by taking into account the microstructure curvature tensor as well as different transformation properties of polar and axial tensors. Our material symmetry group consists of ordered triples of tensors which make the strain energy density of the polar-elastic continuum invariant under change of the reference placement. Within the polar-elastic solids we discuss the isotropic, hemitropic, orthotropic, transversely isotropic and cubic-symmetric materials and give explicitly the consistently reduced representations of the strain energy density.

1 Introduction

Mechanics of Micropolar Continua (also called Cosserat Continua or Polar Continua) was first summarized in 1909 by the Cosserat brothers in their centurial book [1] but without consideration of the constitutive equations. In the books by Eringen [4, 5], Nowacki [8] and Eremeyev et al. [2] various constitutive equations of the micropolar elastic continuum were considered and widely discussed. The Cosserat continuum model is frequently used for description of complex media such as composites, foams, cellular solids, lattices, masonries, particle assemblies, mag-

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netic rheological fluids, liquid crystals, etc. For characterizations of material behaviour of micropolar continua a great role plays the material symmetry group. The group for the non-linear micropolar continuum was first characterized by Eringen and Kafadar [6]. They discussed all density-preserving deformations and all microrotations of the reference placement of the micropolar continuum that cannot be experimentally detected. In terms of members of the group definitions of the simple micropolar solid and the simple micropolar fluid were given.

In [3] we extended the definition of the material symmetry group proposed by Eringen and Kafadar [6]. We considered the polar-elastic material characterized by the strain energy density W and introduced the following modifications:

1. At each material point the strain energy density W, satisfying the principle of material frame-indifference, depends explicitly not only on the natural Lagrangian stretch $\mathbf{E}$ and wryness $\mathbf{\Gamma}$ tensors, but additionally upon the microstructure curvature tensor $\mathbf{B}$ of the undeformed placement as the parametric tensor. The necessity of using these three fields in W was shown in [9]. The tensor $\mathbf{B}$ appears naturally during change of the reference placement. The case $\mathbf{B} \neq \mathbf{0}$ corresponds to non-uniform distribution of directors in the reference placement. In [6] the similar strain measures were used in W, but the referential mass density $\rho_\varkappa$ and the microinertia tensor $\mathbf{J}_\varkappa$ were introduced as the parametric quantities in W.
2. Considering invariance properties of W we take into account that $\mathbf{E}$ is the polar tensor, but $\mathbf{\Gamma}$ and $\mathbf{B}$ are the axial tensors which change their signs under inversion transformation (mirror reflection) of 3D space. Eringen and Kafadar [6] did not take into account that their $\mathbf{\Gamma}$ was the axial tensor. As a result, difference between the orthogonal tensors and the proper orthogonal tensors considered as members of our material symmetry group leads to additional essential reduction of W.
3. Our material symmetry group $\mathscr{G}_\varkappa$ consists of the ordered triple of tensors: the unimodular $\mathbf{P}$, the orthogonal $\mathbf{R}$, and the second-order $\mathbf{L}$ one. These tensors appear from transformation of $\mathbf{E}, \mathbf{\Gamma}$ and $\mathbf{B}$ under an arbitrary change of the reference placements of the micropolar body. The transformation properties of $\mathbf{B}$ are quite different from those of $\mathbf{J}_\varkappa$.

As a result of these modifications, the material symmetry group $\mathscr{G}_\varkappa$ in [3] does not coincide with the group introduced in [6].

In this paper we consider the consistently reduced constitutive equations of the non-linear anisotropic elastic micropolar solids. In addition to [3] we present the lists of additional joint invariants of W describing the orthotropic and transversely isotropic micropolar solids.

2 Basic Relations of the Cosserat Continuum

Let the micropolar body $\mathscr{B}$ deform in the three-dimensional (3D) Euclidean physical space $\mathscr{E}$ which translation vector space is E. The finite deformation of the polar-elastic body $\mathscr{B}$ can be described by mapping from the reference (undeformed) placement $\varkappa(\mathscr{B}) = B_\varkappa \subset \mathscr{E}$ to the actual (deformed) placement $\gamma(\mathscr{B}) = B_\gamma = \chi(B_\varkappa) \in \mathscr{E}$.

In $\varkappa(\mathscr{B})$ the position $x \in \mathscr{E}$ of the material particle $X \in \mathscr{B}$ is given by the vector $\mathbf{x} \in E$ relative to the origin $o \in \mathscr{E}$ of an inertial frame $(o, \mathbf{i}_a)$, where $\mathbf{i}_a \in E$, $a = 1,2,3$, is a right-handed triple of orthonormal vectors. Orientation of $X \in \mathscr{B}$ in E is fixed by the right-handed triple of orthonormal directors $\mathbf{h}_a \in E$.

In $\gamma(\mathscr{B})$, $\chi = \gamma \circ \varkappa^{-1}$, the position $y \in B_\varkappa$ of the same material particle $X \in \mathscr{B}$ becomes defined by the vector $\mathbf{y} \in E$ taken here relative to the same origin $o \in \mathscr{E}$. The orientation of X becomes fixed by the right-handed triple of orthonormal directors $\mathbf{d}_a \in E$.

As a result, the finite deformation of the polar-elastic body is described by the following two smooth mappings:

$$\mathbf{y} = \chi(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}), \qquad \mathbf{d}_a = \mathbf{Q}(\mathbf{x})\mathbf{h}_a, \tag{1}$$

where $\mathbf{u} \in E$ is the translation vector and $\mathbf{Q} = \mathbf{d}_a \otimes \mathbf{h}_a \in Orth^+$ is the proper orthogonal microrotation tensor, $\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det \mathbf{Q} = +1$. Two independent fields $\mathbf{u}(\mathbf{x})$ and $\mathbf{Q}(\mathbf{x})$ describe translational and rotational degrees of freedom of the polar-elastic continuum.

The natural Lagrangian relative stretch and wryness (or change of the microstructure orientation) tensors $\mathbf{E}$ and $\mathbf{\Gamma}$ are defined according to [9] as

$$\mathbf{E} = \mathbf{Q}^T\mathbf{F} - \mathbf{I}, \qquad \mathbf{\Gamma} = -\frac{1}{2}\mathsf{E} : (\mathbf{Q}^T \operatorname{Grad} \mathbf{Q}). \tag{2}$$

Here $\mathbf{F} = \operatorname{Grad} \mathbf{y}$, $\det \mathbf{F} > 0$, is the classical deformation gradient tensor taken relative to $B_\varkappa$, $\mathbf{I}$ is the identity (metric) tensor of the space E, $\mathsf{E} = -\mathbf{I} \times \mathbf{I}$ is the 3rd-order skew permutation tensor with $\times$ the vector product, while the double dot product $:$ of two 3rd-order tensors A, B represented in the base $\mathbf{h}_a$ is defined as $\mathsf{A} : \mathsf{B} = A_{amn}B_{mnb}\mathbf{h}_a \otimes \mathbf{h}_b$.

The wryness tensor $\mathbf{\Gamma}$ can also be expressed in the equivalent forms, see [9],

$$\mathbf{\Gamma} = -\frac{1}{2}\mathbf{h}_a \times (\mathbf{h}_a \mathbf{Q}^T \operatorname{Grad} \mathbf{Q}) = \mathbf{Q}^T\mathbf{C}\mathbf{F} - \mathbf{B}, \tag{3}$$

where $\mathbf{B}$ and $\mathbf{C}$ are the respective microstructure curvature tensors of the polar continuum in the reference and actual placements defined by

$$\mathbf{B} = \frac{1}{2}\mathbf{h}_a \times \operatorname{Grad} \mathbf{h}_a, \qquad \mathbf{C} = \frac{1}{2}\mathbf{d}_a \times \operatorname{grad} \mathbf{d}_a, \tag{4}$$

with the operator grad being taken in the deformed placement B_γ.

In what follows $\mathbf{B}$ and $\mathbf{C}$ play an important role because they characterize the non-uniform distribution of directors $\mathbf{h}_a$ and $\mathbf{d}_a$ in the reference and actual placements, respectively. In particular, if $\mathbf{h}_a$ are constant in space then $\mathbf{B} = \mathbf{0}$. Tensors $\mathbf{B}$ and $\mathbf{C}$ can be used instead of $\mathbf{h}_a$ and $\mathbf{d}_a$ as primary quantities. Indeed, $\mathbf{h}_a$ and $\mathbf{d}_a$ can be found from $\mathbf{B}$ and $\mathbf{C}$, respectively, if some compatibility conditions in terms of $\mathbf{B}$

and $\mathbf{C}$ are fulfilled. The compatibility condition for $\mathbf{B}$ follows from

$$\mathbf{b}_{k,s} = \mathbf{b}_{s,k} + \mathbf{b}_s \times \mathbf{b}_k, \tag{5}$$

where $\mathbf{b}_k = \mathbf{B}\mathbf{i}_k$, and indices after comma denote differentiation with respect to Cartesian coordinates in the reference placement x_1, x_2, x_3, for example $\mathbf{b}_{s,k} = \frac{\partial \mathbf{B}_s}{\partial x_k}$. The compatibility condition for $\mathbf{C}$ follows from relation similar to (5) but the vectors $\mathbf{c}_k = \mathbf{C}\mathbf{i}_k$ are differentiated with respect to Cartesian coordinates in the actual placement y_s.

The material behaviour of the micropolar (hyper)elastic continuum is described by the strain energy density $W_\varkappa$ per unit volume of the undeformed placement $B_\varkappa$. The density $W_\varkappa$ satisfying the principle of material frame-indifference takes the reduced form

$$W_\varkappa = \widehat{W}_\varkappa(\mathbf{E}, \boldsymbol{\Gamma}; \mathbf{x}, \mathbf{B}). \tag{6}$$

We call the polar-elastic continuum homogeneous if there exists a reference placement $B_\varkappa$ such that $W_\varkappa$ does not depend on $\mathbf{x}$ and materially uniform if $W_\varkappa$ does not depend on $\mathbf{B}$ or $\mathbf{B} \equiv \mathbf{0}$.

Definition of the material symmetry group is based on invariance of $W_\varkappa$ under change of the reference placement. Let us introduce another reference placement $\varkappa_*(\mathscr{B}) = B_* \in \mathscr{E}$ of $\mathscr{B}$, in which the position $x_* \in B_*$ of $X \in \mathscr{B}$ is given by the vector $\mathbf{x}_*$ relative to the same origin $o \in \mathscr{E}$ and its orientation is fixed by three orthonormal directors $\mathbf{h}_{*a}$. Let $\mathbf{P} = \operatorname{Grad} \mathbf{x}_*$, $\det \mathbf{P} \neq 0$, be the deformation gradient tensor transforming $d\mathbf{x}$ into $d\mathbf{x}_*$, and $\mathbf{R} \in Orth$ be the rotation tensor transforming $\mathbf{h}_a$ into $\mathbf{h}_{*a}$, so that

$$d\mathbf{x}_* = \mathbf{P} d\mathbf{x}, \quad \mathbf{h}_{*a} = \mathbf{R}\mathbf{h}_a. \tag{7}$$

In what follows all fields associated with deformation relative to the reference placement B_* will be marked by the lower index $*$. We obtain the following transformation relations, see [3] for details:

$$\mathbf{F} = \mathbf{F}_* \mathbf{P}, \quad \mathbf{Q} = \mathbf{Q}_* \mathbf{R}, \tag{8}$$

$$\begin{aligned} \mathbf{E}_* = \mathbf{Q}_*^T \mathbf{F}_* - \mathbf{I} &= \mathbf{R}\mathbf{E}\mathbf{P}^{-1} + \mathbf{R}\mathbf{P}^{-1} - \mathbf{I} \\ &= \mathbf{R}(\mathbf{E} + \mathbf{I})\mathbf{P}^{-1} - \mathbf{I}, \end{aligned} \tag{9}$$

$$\mathbf{B}_* = (\det \mathbf{R}) \mathbf{R}\mathbf{B}\mathbf{P}^{-1} - \mathbf{L}, \qquad \boldsymbol{\Gamma}_* = (\det \mathbf{R}) \mathbf{R}\boldsymbol{\Gamma}\mathbf{P}^{-1} + \mathbf{L}, \tag{10}$$

where

$$\mathbf{L} = \mathbf{R}\mathbf{Z}\mathbf{P}^{-1}, \quad \mathbf{Z} = -\frac{1}{2} \mathsf{E} : (\mathbf{R} \operatorname{Grad} \mathbf{R}^T). \tag{11}$$

Let us note that the form of elastic strain energy density $W_\varkappa$ of the micropolar body at any particle $X \in \mathscr{B}$ depends upon the choice of the reference placement, in general. Particularly important are sets of reference placements which leave unchanged the form of the energy density. Transformations of the reference placement under which the energy density remains unchanged we call here invariant transformations. Knowledge of all such invariant transformations allows one to precisely define the fluid, the solid, the liquid crystal or the subfluid as well as to introduce notions of isotropic or anisotropic hyper-elastic continua. Similar approach is used in classical continuum mechanics and in non-linear elasticity in [14, 15].

The elastic strain energy density W_* relative to the changed reference placement B_* depends in each point $x_* \in B_\star$ on the stretch tensor $\mathbf{E}_*$, the wryness tensor $\boldsymbol{\Gamma}_*$, and also upon the structure curvature tensor $\mathbf{B}_*$. This dependence may, in general, be different than that of $W_\varkappa(\mathbf{E},\boldsymbol{\Gamma};\mathbf{x},\mathbf{B})$. However, the strain energy of any part of the polar-elastic continuum should be conserved, so that

$$\int\limits_{P_\varkappa} W_\varkappa \, dv_\varkappa = \int\limits_{P_*} W_* \, dv_* \tag{12}$$

for any part of the micropolar body $P_\varkappa \subset B_\varkappa$ corresponding to $P_* \subset B_*$, because the functions $W_\varkappa$ and W_* describe the strain energy density of the same deformed state of $P_\gamma \subset B_\gamma = \chi(P_\varkappa) = \chi_*(P_*)$, where χ_* is the deformation function from B_* to B_γ.

Changing variables $\mathbf{x}_* \to \mathbf{x}$ in the right-hand integral of (12) we obtain

$$\int\limits_{P_*} W_*[\mathbf{E}_*(\mathbf{x}_*),\boldsymbol{\Gamma}_*(\mathbf{x}_*);\mathbf{x}_*,\mathbf{B}_*(\mathbf{x}_*)]\, dv_* = \int\limits_{P_\varkappa} |\det\mathbf{P}|\, W_*[\mathbf{E}_*(\mathbf{x}),\boldsymbol{\Gamma}_*(\mathbf{x});\mathbf{x},\mathbf{B}_*(\mathbf{x})]\, dv_\varkappa .$$

Thus, from (12) it follows that W_* and $W_\varkappa$ are related by

$$|\det\mathbf{P}|\, W_*(\mathbf{E}_*,\boldsymbol{\Gamma}_*;\mathbf{x},\mathbf{B}_*) = W_\varkappa(\mathbf{E},\boldsymbol{\Gamma};\mathbf{x},\mathbf{B}).$$

Here $\mathbf{E}_*$, $\boldsymbol{\Gamma}_*$, and $\mathbf{B}_*$ are expressed as in (9) and (10).

From physical reasons invariant transformations of the reference placement should preserve the elementary volume of $B_\varkappa$. Hence, the tensor $\mathbf{P}$ should belong to the unimodular group for which $|\det\mathbf{P}| = 1$.

The assumption that the constitutive relation is insensitive to the change of the reference placement means that the explicit forms of the strain energy densities $W_\varkappa$ and W_* should coincide, that is

$$W_\varkappa(\mathbf{E},\boldsymbol{\Gamma};\mathbf{x},\mathbf{B}) = W_\varkappa(\mathbf{E}_*,\mathbf{K}_*;\mathbf{x},\mathbf{B}_*).$$

In other words, this means that one may use the same function for the strain energy density independently on the choice of $B_\varkappa$ or B_*, but with different expressions for stretch and wryness tensors as well as for the microstructure curvature tensor. In what follows we not always explicitly indicate that all the functions depend also on the position vector $\mathbf{x}$ and W is taken relative to the undeformed placement $B_\varkappa$.

Using (9) and (10) we obtain the following invariance requirement for W under change of the reference placement:

$$W(\mathbf{E},\boldsymbol{\Gamma};\mathbf{B}) = W\left[\mathbf{REP}^{-1} + \mathbf{RP}^{-1} - \mathbf{I},\ (\det\mathbf{R})\mathbf{R}\boldsymbol{\Gamma}\mathbf{P}^{-1} + \mathbf{L};\ (\det\mathbf{R})\mathbf{RBP}^{-1} - \mathbf{L}\right]. \tag{13}$$

The relation (13) holds locally, i.e. it should be satisfied at any x and $\mathbf{B}$, and the tensors $\mathbf{P}$, $\mathbf{R}$, $\mathbf{L}$ are treated as independent here. As a result, the local invariance of W under change of the reference placement is described by the triple of tensors $(\mathbf{P},\mathbf{R},\mathbf{L})$.

In what follows we use the following nomenclature:
$Orth = \{\mathbf{O} : \mathbf{O}^{-1} = \mathbf{O}^T, \ \ \det\mathbf{O} = \pm 1\}$ – the group of orthogonal tensors;
$Orth^+ = \{\mathbf{O} : \mathbf{O} \in Orth, \ \ \det\mathbf{O} = 1\}$ – the group of rotation tensors;
$Unim = \{\mathbf{P} : \mathbf{P} \in E \otimes E, \ \ \det\mathbf{P} = \pm 1\}$ – the unimodular group;
$Lin = \{\mathbf{L} \in E \otimes E\}$ – the linear group.

Here *Orth* and *Unim* are groups with regard to multiplication, and *Lin* is the group with regard to addition.

3 Definition of the Material Symmetry Group

Following [3] and using (13) we give the following definition:

Definition 1 *By the material symmetry group* $\mathscr{G}_\varkappa$ *at* x *and* $\mathbf{B}$ *of the polar-elastic continuum we call all sets of ordered triples of tensors*

$$\mathbb{X} = (\mathbf{P} \in Unim, \mathbf{R} \in Orth, \mathbf{L} \in Lin), \tag{14}$$

satisfying the relation

$$W(\mathbf{E},\boldsymbol{\Gamma};\mathbf{B}) = W\left[\mathbf{R}\mathbf{E}\mathbf{P}^{-1} + \mathbf{R}\mathbf{P}^{-1} - \mathbf{I},\ (\det\mathbf{R})\mathbf{R}\boldsymbol{\Gamma}\mathbf{P}^{-1} + \mathbf{L};\ (\det\mathbf{R})\mathbf{R}\mathbf{B}\mathbf{P}^{-1} - \mathbf{L}\right] \tag{15}$$

for any tensors $\mathbf{E}$, $\boldsymbol{\Gamma}$, $\mathbf{B}$ *in domain of definition of the function* W.

The set $\mathscr{G}_\varkappa$ is the group relative to the group operation $\circ$ defined by

$$(\mathbf{P}_1,\mathbf{R}_1,\mathbf{L}_1) \circ (\mathbf{P}_2,\mathbf{R}_2,\mathbf{L}_2) = \left[\mathbf{P}_1\mathbf{P}_2,\ \mathbf{R}_1\mathbf{R}_2,\ \mathbf{L}_1 + (\det\mathbf{R}_1)\mathbf{R}_1\mathbf{L}_2\mathbf{P}_1^{-1}\right].$$

In terms of members of $\mathscr{G}_\varkappa$ the polar-elastic fluids, solids, liquid crystals, and subfluids can be conveniently defined, see [3] for details.

In what follows we restrict ourselves to the polar-elastic solids which are defined as follows:

Definition 2 *The micropolar elastic continuum is called the polar-elastic solid at* x *and* $\mathbf{B}$ *if there exists a reference placement* $B_\varkappa$*, called undistorted, such that the material symmetry group relative to* $B_\varkappa$ *is given by*

$$\mathscr{G}_\varkappa = \mathscr{R}_\varkappa \equiv \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0}) : \quad \mathbf{O} \in \mathscr{O}_\varkappa \subset Orth\}. \tag{16}$$

The group $\mathscr{R}_\varkappa$ is fully described by a subgroup $\mathscr{O}_\varkappa$ of orthogonal group *Orth*. Invariance requirement of W leads here to finding the subgroup $\mathscr{O}_\varkappa$ such that

$$W(\mathbf{E},\mathbf{\Gamma};\mathbf{B}) = W\left[\mathbf{OEO}^T, (\det\mathbf{O})\mathbf{O\Gamma O}^T; (\det\mathbf{O})\mathbf{OBO}^T\right], \quad \forall\mathbf{O}\in\mathscr{O}_\varkappa. \tag{17}$$

4 Consistently Simplified Forms of the Strain Energy Density

Let us discuss consistently simplified forms of W corresponding to some particular cases of anisotropic micropolar solids. We begin from the isotropic material.

Definition 3 ***Isotropic material.*** *The polar-elastic solid is called isotropic at x and* **B** *if there exists a reference placement $B_\varkappa$, called undistorted, such that the material symmetry group relative to $B_\varkappa$ takes the form*

$$\mathscr{G}_\varkappa = \mathscr{S}_\varkappa \equiv \{(\mathbf{P}=\mathbf{O},\mathbf{O},\mathbf{0}): \quad \mathbf{O}\in Orth\}. \tag{18}$$

This definition means that the strain energy density of the polar-elastic isotropic solid satisfies the relation

$$W(\mathbf{E},\mathbf{\Gamma};\mathbf{B}) = W\left[\mathbf{OEO}^T, (\det\mathbf{O})\mathbf{O\Gamma O}^T; (\det\mathbf{O})\mathbf{OBO}^T\right], \quad \forall\mathbf{O}\in Orth.$$

Scalar-valued isotropic functions of a few 2nd-order tensors can be expressed by the so-called representation theorems in terms of joint invariants of the tensorial arguments, called also the integrity basis, see [12, 13]. Decomposing the non-symmetric tensors $\mathbf{E}$, $\mathbf{\Gamma}$ and $\mathbf{B}$ into their symmetric and skew parts,

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_S + \mathbf{E}_A, \quad \mathbf{E}_S = \frac{1}{2}(\mathbf{E}+\mathbf{E}^T), \quad \mathbf{E}_A = \frac{1}{2}(\mathbf{E}-\mathbf{E}^T),\\
\mathbf{\Gamma} &= \mathbf{\Gamma}_S + \mathbf{\Gamma}_A, \quad \mathbf{\Gamma}_S = \frac{1}{2}(\mathbf{\Gamma}+\mathbf{\Gamma}^T), \quad \mathbf{\Gamma}_A = \frac{1}{2}(\mathbf{\Gamma}-\mathbf{G}^T),\\
\mathbf{B} &= \mathbf{B}_S + \mathbf{B}_A, \quad \mathbf{B}_S = \frac{1}{2}(\mathbf{B}+\mathbf{B}^T), \quad \mathbf{B}_A = \frac{1}{2}(\mathbf{B}-\mathbf{B}^T),
\end{aligned}$$

we represent the strain energy density as the function of three symmetric and three skew tensors,

$$W = W(\mathbf{E}_S,\mathbf{E}_A,\mathbf{\Gamma}_S,\mathbf{\Gamma}_A;\mathbf{B}_S,\mathbf{B}_A). \tag{19}$$

The integrity basis for the proper orthogonal group is given by Spencer, see Table 1 in [12] or Table II in [13]. For the proper orthogonal group there is no difference in transformations of the axial and polar tensors. It is not the case if one considers transformations using the full orthogonal group. Since $\mathbf{\Gamma}_S,\mathbf{\Gamma}_A,\mathbf{B}_S,\mathbf{B}_A$ are the axial tensors, not all invariants listed in [12, 13] are absolute invariants under orthogonal transformations, because some of them change sign under non-proper orthogonal transformations. Such invariants are called relative invariants [13]. Examples of relative invariants are $\operatorname{tr}\mathbf{\Gamma}_S$, $\operatorname{tr}\mathbf{\Gamma}_S^3$, $\operatorname{tr}\mathbf{E}_S\mathbf{\Gamma}_S$, $\operatorname{tr}\mathbf{E}_S\mathbf{B}_S$, etc. This gives us the following property of W:

$$W(\mathbf{E}_S,\mathbf{E}_A,\mathbf{\Gamma}_S,\mathbf{\Gamma}_A;\mathbf{B}_S,\mathbf{B}_A) = W(\mathbf{E}_S,\mathbf{E}_A,-\mathbf{\Gamma}_S,-\mathbf{\Gamma}_A;-\mathbf{B}_S,-\mathbf{B}_A). \tag{20}$$

Using the representations given by Zheng [16], we present the lists of absolute and relative polynomial invariants for the polar-elastic isotropic solid in Table 1. In this case there are 119 invariants. They constitute so-called irreducible integrity basis. The strain energy density of the polar-elastic isotropic solid is given by any scalar-valued function of these invariants satisfying (20).

Further simplifications are possible if we neglect the explicit dependence of W on $\mathbf{B}$, that is if we assume that $W = W(\mathbf{E},\mathbf{\Gamma})$. The integrity basis of two non-symmentric tensors under the orthogonal group contains 39 members, see Ramezani et al. [10] where these invariants are listed and the corresponding constitutive equations are proposed. Kafadar and Eringen [7] constructed the list of independent invariants. Table 1 contains the invariants of [10] and of [7] as well as aditional joint invariants of $\mathbf{E}$, $\mathbf{\Gamma}$ and $\mathbf{B}$. According to [7], the isotropic scalar-valued function $W = W(\mathbf{E},\mathbf{\Gamma})$ is expressible in terms of 15 invariants,

$$W = W(I_1, I_2, \ldots, I_{15}), \tag{21}$$

where I_k are given by

$$\begin{aligned}
I_1 &= \operatorname{tr}\mathbf{E}, & I_2 &= \operatorname{tr}\mathbf{E}^2, & I_3 &= \operatorname{tr}\mathbf{E}^3,\\
I_4 &= \operatorname{tr}\mathbf{E}\mathbf{E}^T, & I_5 &= \operatorname{tr}\mathbf{E}^2\mathbf{E}^T, & I_6 &= \operatorname{tr}\mathbf{E}^2\mathbf{E}^{T2},\\
I_7 &= \operatorname{tr}\mathbf{E}\mathbf{\Gamma}, & I_8 &= \operatorname{tr}\mathbf{E}^2\mathbf{\Gamma}, & I_9 &= \operatorname{tr}\mathbf{E}\mathbf{\Gamma}^2,\\
I_{10} &= \operatorname{tr}\mathbf{\Gamma}, & I_{11} &= \operatorname{tr}\mathbf{\Gamma}^2, & I_{12} &= \operatorname{tr}\mathbf{\Gamma}^3,\\
I_{13} &= \operatorname{tr}\mathbf{\Gamma}\mathbf{\Gamma}^T, & I_{14} &= \operatorname{tr}\mathbf{\Gamma}^2\mathbf{\Gamma}^T, & I_{15} &= \operatorname{tr}\mathbf{\Gamma}^2\mathbf{\Gamma}^{T2}.
\end{aligned}$$

Taking into account that $W = W(\mathbf{E},\mathbf{\Gamma})$ is an even function with respect to $\mathbf{\Gamma}$, because in our case the group $\mathscr{S}_\varkappa$ contains the reflection $-\mathbb{I}$, W becomes also the even function with respect to some invariants,

$$\begin{aligned}
&W(I_1,I_2,I_3,I_4,I_5,I_6,I_7,I_8,I_9,I_{10},I_{11},I_{12},I_{13},I_{14},I_{15})\\
&\quad = W(I_1,I_2,I_3,I_4,I_5,I_6,-I_7,-I_8,I_9,-I_{10},I_{11},-I_{12},I_{13},-I_{14},I_{15}).
\end{aligned} \tag{22}$$

Expanding W into the Taylor series relative to $\mathbf{E}$ and $\mathbf{\Gamma}$, and keeping up to quadratic terms, we obtain the approximate polynomial representation of (22),

$$\begin{aligned}
W =& w_0 + a_1 I_1 + b_1 I_1^2 + b_3 I_{10}^2 + b_4 I_4 + b_5 I_2 + b_7 I_{11} + b_8 I_{13}\\
&+ O(\max(\|\mathbf{E}\|^3, \|\mathbf{\Gamma}\|^3)),
\end{aligned} \tag{23}$$

where w_0, a_1, $b_1, \ldots, b_8$ are material constants.

We may also consider the representation of W which takes the form of sum of two scalar functions each depending on one strain measure,

$$W = W_1(\mathbf{E}) + W_2(\mathbf{\Gamma}). \tag{24}$$

The form (24) was used for example in [10] in order to generalize the classical neo-Hookean and Moony-Rivlin models to the polar-elastic solids. Using [11] we obtain the following representation of W:

Table 1 119 invariants in W in the case of polar-elastic isotropic solid

Agencies	Invariants			
$\mathbf{E}_S$	$\mathrm{tr}\,\mathbf{E}_S$	$\mathrm{tr}\,\mathbf{E}_S^2$	$\mathrm{tr}\,\mathbf{E}_S^3$	
$\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_A^2$			
$\mathbf{E}_S$, $\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A^2$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{E}_A^2$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{E}_A^2\mathbf{E}_S\mathbf{E}_A$	
$\boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^3$	
$\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_A^2$			
$\boldsymbol{\Gamma}_S$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A^2$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\boldsymbol{\Gamma}_A^2$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\boldsymbol{\Gamma}_A^2\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A$	
$\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{B}_S^2$	$\mathrm{tr}\,\mathbf{B}_S^3$	
$\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{B}_A^2$			
$\mathbf{B}_S$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{B}_S\mathbf{B}_A^2$	$\mathrm{tr}\,\mathbf{B}_S^2\mathbf{B}_A^2$	$\mathrm{tr}\,\mathbf{B}_S^2\mathbf{B}_A^2\mathbf{B}_S\mathbf{B}_A$	
$\mathbf{E}_S$, $\boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\mathbf{E}_S^2\boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S^2$	$\mathrm{tr}\,\mathbf{E}_S^2\boldsymbol{\Gamma}_S^2$
$\mathbf{E}_S$, $\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_S^2$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{B}_S^2$
$\boldsymbol{\Gamma}_S$, $\mathbf{B}_S$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_S$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{B}_S$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_S^2$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{B}_S^2$
$\mathbf{E}_S$, $\boldsymbol{\Gamma}_S$, $\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S\mathbf{B}_S$			
$\mathbf{E}_A$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_A\boldsymbol{\Gamma}_A$			
$\mathbf{E}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_A\mathbf{B}_A$			
$\boldsymbol{\Gamma}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_A\mathbf{B}_A$			
$\mathbf{E}_A$, $\boldsymbol{\Gamma}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_A\boldsymbol{\Gamma}_A\mathbf{B}_A$			
$\mathbf{E}_S$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_A^2$	$\mathrm{tr}\,\mathbf{E}_S^2\boldsymbol{\Gamma}_A^2$	$\mathrm{tr}\,\mathbf{E}_S^2\boldsymbol{\Gamma}_A^2\mathbf{E}_S\boldsymbol{\Gamma}_A$	
$\mathbf{E}_S$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_A^2$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{B}_A^2$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{B}_A^2\mathbf{E}_S\mathbf{B}_A$	
$\boldsymbol{\Gamma}_S$, $\mathbf{E}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{E}_A^2$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{E}_A^2$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{E}_A^2\boldsymbol{\Gamma}_S\mathbf{E}_A$	
$\boldsymbol{\Gamma}_S$, $\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_A^2$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{B}_A^2$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{B}_A^2\boldsymbol{\Gamma}_S\mathbf{B}_A$	
$\mathbf{B}_S$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{B}_S\boldsymbol{\Gamma}_A^2$	$\mathrm{tr}\,\mathbf{B}_S^2\boldsymbol{\Gamma}_A^2$	$\mathrm{tr}\,\mathbf{E}_S^2\boldsymbol{\Gamma}_A^2\mathbf{B}_S\boldsymbol{\Gamma}_A$	
$\mathbf{B}_S$, $\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{B}_S\mathbf{E}_A^2$	$\mathrm{tr}\,\mathbf{B}_S^2\mathbf{E}_A^2$	$\mathrm{tr}\,\mathbf{B}_S^2\mathbf{E}_A^2\mathbf{B}_S\mathbf{E}_A$	
$\mathbf{E}_S$, $\boldsymbol{\Gamma}_S$, $\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S^2\boldsymbol{\Gamma}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S^2\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A^2\boldsymbol{\Gamma}_S\mathbf{E}_A$
$\mathbf{E}_S$, $\boldsymbol{\Gamma}_S$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S^2\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S^2\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_A^2\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A$
$\mathbf{E}_S$, $\boldsymbol{\Gamma}_S$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S^2\boldsymbol{\Gamma}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_S^2\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_A^2\boldsymbol{\Gamma}_S\mathbf{B}_A$
$\mathbf{E}_S$, $\mathbf{B}_S$, $\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{B}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_S^2\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A^2\mathbf{B}_S\mathbf{E}_A$
$\mathbf{E}_S$, $\mathbf{B}_S$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{B}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_S^2\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_A^2\mathbf{B}_S\boldsymbol{\Gamma}_A$
$\mathbf{E}_S$, $\mathbf{B}_S$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S^2\mathbf{B}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_S^2\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{B}_A^2\mathbf{B}_S\mathbf{B}_A$
$\boldsymbol{\Gamma}_S$, $\mathbf{B}_S$, $\mathbf{E}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_S\mathbf{E}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{B}_S\mathbf{E}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_S^2\mathbf{E}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{E}_A^2\mathbf{B}_S\mathbf{E}_A$
$\boldsymbol{\Gamma}_S$, $\mathbf{B}_S$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{B}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_S^2\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A^2\mathbf{B}_S\boldsymbol{\Gamma}_A$
$\boldsymbol{\Gamma}_S$, $\mathbf{B}_S$, $\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_S\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S^2\mathbf{B}_S\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_S^2\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{B}_A^2\mathbf{B}_S\mathbf{B}_A$
$\mathbf{E}_S$, $\mathbf{E}_A$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A^2\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A\boldsymbol{\Gamma}_A^2$	
$\mathbf{E}_S$, $\mathbf{E}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A^2\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\mathbf{E}_A\mathbf{B}_A^2$	
$\mathbf{E}_S$, $\boldsymbol{\Gamma}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_A\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_A^2\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{E}_S\boldsymbol{\Gamma}_A\mathbf{B}_A^2$	
$\boldsymbol{\Gamma}_S$, $\mathbf{E}_A$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{E}_A\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{E}_A^2\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{E}_A\boldsymbol{\Gamma}_A^2$	
$\boldsymbol{\Gamma}_S$, $\mathbf{E}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{E}_A\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{E}_A^2\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\mathbf{E}_A\mathbf{B}_A^2$	
$\boldsymbol{\Gamma}_S$, $\boldsymbol{\Gamma}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A^2\mathbf{B}_A$	$\mathrm{tr}\,\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A\mathbf{B}_A^2$	
$\mathbf{B}_S$, $\mathbf{E}_A$, $\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{B}_S\mathbf{E}_A\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{B}_S\mathbf{E}_A^2\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{B}_S\mathbf{E}_A\boldsymbol{\Gamma}_A^2$	
$\mathbf{B}_S$, $\mathbf{E}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{B}_S\mathbf{E}_A\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{B}_S\mathbf{E}_A^2\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{B}_S\mathbf{E}_A\mathbf{B}_A^2$	
$\mathbf{B}_S$, $\boldsymbol{\Gamma}_A$, $\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{B}_S\boldsymbol{\Gamma}_A\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{B}_S\boldsymbol{\Gamma}_A^2\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{B}_S\boldsymbol{\Gamma}_A\mathbf{B}_A^2$	

$$W = \widetilde{W}_1(I_1, \ldots, I_6) + \widetilde{W}_2(I_{10}, \ldots, I_{15}), \tag{25}$$

where $\widetilde{W}_2$ has the property

$$\widetilde{W}_2(I_{10},I_{11},I_{12},I_{13},I_{14},I_{15}) = \widetilde{W}_2(-I_{10},I_{11},-I_{12},I_{13},-I_{14},I_{15}). \tag{26}$$

Expanding (25) with (26) into the Taylor series and keeping up to quadratic terms in $\mathbf{E}$ and $\boldsymbol{\Gamma}$, W takes the form (24) with

$$W_1 = w_0 + a_1 I_1 + b_1 I_1^2 + b_4 I_4 + b_5 I_2, \quad W_2 = b_3 I_{10}^2 + b_7 I_{11} + b_8 I_{13}.$$

If in the definition (18) we use only the proper orthogonal tensors then the resulting constitutive equations correspond to the hemitropic polar-elastic continuum.

Definition 4 ***Hemitropic material.*** *The polar-elastic solid is called hemitropic at x and* $\mathbf{B}$ *if there exists a reference placement* $B_\varkappa$*, called undistorted, such that the material symmetry group relative to* $B_\varkappa$ *takes the form*

$$\mathscr{G}_\varkappa = \mathscr{S}_\varkappa^+ \equiv \left\{ (\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0}) : \quad \mathbf{O} \in Orth^+ \right\}. \tag{27}$$

The strain energy density of the hemitropic polar-elastic solid satisfies the relation

$$W(\mathbf{E}, \boldsymbol{\Gamma}; \mathbf{B}) = W\left(\mathbf{O}\mathbf{E}\mathbf{O}^T, \mathbf{O}\boldsymbol{\Gamma}\mathbf{O}^T; \mathbf{O}\mathbf{B}\mathbf{O}^T\right), \quad \forall \mathbf{O} \in Orth^+. \tag{28}$$

The hemitropic polar-elastic solid is insensitive to the change of orientation of the space. In the case of reduced strain energy density $W = W(\mathbf{E}, \boldsymbol{\Gamma})$ the representation of W is given by (21), but the property (22) does not hold, in general. Obviously, the polar-elastic isotropic solid is also hemitropic.

Definitions (18) and (28) are somewhat similar to the corresponding definition of the isotropic polar-elastic solid proposed by Eringen and Kafadar [6]. However, the properties (22) or (26) do not follow from the definition used in [6].

Definition 5 ***Orthotropic material.*** *The polar-elastic solid is called orthotropic at x and* $\mathbf{B}$ *if the material symmetry group for some reference placement* $B_\varkappa$ *takes the form*

$$\mathscr{G}_\varkappa = \left\{ (\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0}) \right\} : \mathbf{O} = \{ \mathbf{I}, -\mathbf{I}, 2\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{I}, 2\mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{I}, 2\mathbf{e}_3 \otimes \mathbf{e}_3 - \mathbf{I} \}, \tag{29}$$

where $\mathbf{O}$ *are orthogonal tensors performing the mirror reflections and rotations of 180° about three orthonormal vectors* $\mathbf{e}_k$*.*

Obviously, the polar-elastic isotropic solid is also orthotropic. Thus, the invariants given in Tables 1 enter the representation of the strain energy density of the polar-elastic orthotropic solid. The additional list of 60 absolute and relative invariants for the polar-elastic orthotropic solid, which are responsible for the orthotropic properties, is presented in Table 2. Therefore, the full list of Tables 1 and 2 contains 179 invariants.

Definition 6 ***Transversely isotropic solid.*** *The polar-elastic solid is called transversely isotropic at x and* $\mathbf{B}$ *with respect to a direction described by* $\mathbf{e}$ *if the material symmetry group for some reference placement* $B_\varkappa$ *takes the form*

Table 2 Additional 60 invariants in W in the case of polar-elastic orthotropic solid

Agencies	Invariants			
$\mathbf{E}_S$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{E}_S$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S^2$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{E}_S^2$
$\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_A^2$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{E}_A^2$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{E}_A^2\mathbf{V}\mathbf{E}_A$	
$\mathbf{E}_S, \mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{E}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S^2\mathbf{E}_A$	
$\boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\mathbf{V}^2\boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_S^2$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{G}_S^2$
$\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_A^2$	$\mathrm{tr}\,\mathbf{V}^2\boldsymbol{\Gamma}_A^2$	$\mathrm{tr}\,\mathbf{V}^2\boldsymbol{\Gamma}_A^2\mathbf{V}\boldsymbol{\Gamma}_A$	
$\boldsymbol{\Gamma}_S, \boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}^2\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_S^2\boldsymbol{\Gamma}_A$	
$\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{B}_S$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_S^2$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{G}_S^2$
$\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_A^2$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{B}_A^2$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{B}_A^2\mathbf{V}\mathbf{B}_A$	
$\mathbf{B}_S, \mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{B}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_S^2\mathbf{B}_A$	
$\mathbf{E}_S, \boldsymbol{\Gamma}_S$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S\boldsymbol{\Gamma}_S$			
$\mathbf{E}_S, \mathbf{B}_S$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S\mathbf{B}_S$			
$\boldsymbol{\Gamma}_S, \mathbf{B}_S$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_S\mathbf{B}_S$			
$\mathbf{E}_A, \boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_A\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_A^2\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_A\boldsymbol{\Gamma}_A^2$	
$\mathbf{E}_A, \mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_A\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_A^2\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_A\mathbf{B}_A^2$	
$\boldsymbol{\Gamma}_A, \mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_A\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_A^2\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_A\mathbf{B}_A^2$	
$\mathbf{E}_S, \boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{E}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S^2\boldsymbol{\Gamma}_A$	
$\mathbf{E}_S, \mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{E}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{E}_S^2\mathbf{B}_A$	
$\boldsymbol{\Gamma}_S, \mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}^2\boldsymbol{\Gamma}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{G}_S^2\mathbf{B}_A$	
$\boldsymbol{\Gamma}_S, \mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\boldsymbol{\Gamma}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}^2\boldsymbol{\Gamma}_S\mathbf{B}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{G}_S^2\mathbf{B}_A$	
$\mathbf{B}_S, \boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{B}_S\boldsymbol{\Gamma}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_S^2\boldsymbol{\Gamma}_A$	
$\mathbf{B}_S, \mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}^2\mathbf{B}_S\mathbf{E}_A$	$\mathrm{tr}\,\mathbf{V}\mathbf{B}_S^2\mathbf{E}_A$	

$$\mathscr{G}_\varkappa = \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0})\} : \qquad \mathbf{O} = \{\mathbf{I}, -\mathbf{I}, \mathbf{O}(\varphi\mathbf{e}), \quad \forall\, \varphi\}, \tag{30}$$

where $\mathbf{O}(\varphi\mathbf{e}) = (\mathbf{I} - \mathbf{e}\otimes\mathbf{e})\cos\varphi + \mathbf{e}\otimes\mathbf{e} + \mathbf{e}\times\mathbf{I}\sin\varphi$ *is the rotation tensor with the rotation angle* φ *about the unit vector* $\mathbf{e}$*.*

167 invariants for the polar-elastic transversely isotropic solid are presented in Tables 1 and 3.

Definition 7 ***Cubic symmetry.*** *The polar-elastic solid is called cubic-symmetric at* x *and* $\mathbf{B}$ *if the material symmetry group for some reference placement* $B_\varkappa$ *takes the form*

$$\begin{aligned}\mathscr{G}_\varkappa = \{(\mathbf{P} = \mathbf{O}, \mathbf{O}, \mathbf{0})\} :& \\ \mathbf{O} = \{\mathbf{I}, -\mathbf{I},\ \mathbf{e}_1\otimes\mathbf{e}_1 \mp \mathbf{e}_1\times\mathbf{I},\ \mathbf{e}_2\otimes\mathbf{e}_2 \mp \mathbf{e}_2\times\mathbf{I},\ \mathbf{e}_3\otimes\mathbf{e}_3 \mp \mathbf{e}_3\times\mathbf{I}\},&\end{aligned} \tag{31}$$

where $\mathbf{O}$ *are orthogonal tensors performing the mirror reflections and rotations of 90° about three orthonormal vectors* $\mathbf{e}_k$*.*

Here we have discussed the structure of the strain energy density of micropolar elastic solids under finite deformations. Within the linear micropolar elasticity the explicit structure of stiffness tensors was presented is [17] for 14 symmetry groups, see [4].

Table 3 Additional 48 invariants in W in the case of polar-elastic transverse isotropic solid

Agencies	Invariants		
$\mathbf{E}_S$	$\mathbf{e}\cdot\mathbf{E}_S\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_S^2\mathbf{e}$	
$\mathbf{E}_A$	$\mathbf{e}\cdot\mathbf{E}_A^2\mathbf{e}$		
$\mathbf{E}_S$, $\mathbf{E}_A$	$\mathbf{e}\cdot\mathbf{E}_S\mathbf{E}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_S^2\mathbf{E}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_A\mathbf{E}_S\mathbf{E}_A^2\mathbf{e}$
$\boldsymbol{\Gamma}_S$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S^2\mathbf{e}$	
$\boldsymbol{\Gamma}_A$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_A^2\mathbf{e}$		
$\boldsymbol{\Gamma}_S, \boldsymbol{\Gamma}_A$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S^2\boldsymbol{\Gamma}_A\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_A\boldsymbol{\Gamma}_S\boldsymbol{\Gamma}_A^2\mathbf{e}$
$\mathbf{B}_S$	$\mathbf{e}\cdot\mathbf{B}_S\mathbf{e}$	$\mathbf{e}\cdot\mathbf{B}_S^2\mathbf{e}$	
$\mathbf{B}_A$	$\mathbf{e}\cdot\mathbf{B}_A^2\mathbf{e}$		
$\mathbf{B}_S$, $\mathbf{B}_A$	$\mathbf{e}\cdot\mathbf{B}_S\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{B}_S^2\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{B}_A\mathbf{B}_S\mathbf{B}_A^2\mathbf{e}$
$\mathbf{E}_S, \boldsymbol{\Gamma}_S$	$\mathbf{e}\cdot\mathbf{E}_S\boldsymbol{\Gamma}_S\mathbf{e}$		
$\mathbf{E}_S$, $\mathbf{B}_S$	$\mathbf{e}\cdot\mathbf{E}_S\mathbf{B}_S\mathbf{e}$		
$\boldsymbol{\Gamma}_S$, $\mathbf{B}_S$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S\mathbf{B}_S\mathbf{e}$		
$\mathbf{E}_A, \boldsymbol{\Gamma}_A$	$\mathbf{e}\cdot\mathbf{E}_A\boldsymbol{\Gamma}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_A^2\boldsymbol{\Gamma}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_A\boldsymbol{\Gamma}_A^2\mathbf{e}$
$\mathbf{E}_A$, $\mathbf{B}_A$	$\mathbf{e}\cdot\mathbf{E}_A\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_A^2\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_A\mathbf{B}_A^2\mathbf{e}$
$\boldsymbol{\Gamma}_A$, $\mathbf{B}_A$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_A\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_A^2\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_A\mathbf{B}_A^2\mathbf{e}$
$\mathbf{E}_S, \boldsymbol{\Gamma}_A$	$\mathbf{e}\cdot\mathbf{E}_S\boldsymbol{\Gamma}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_S^2\boldsymbol{\Gamma}_A\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_A\mathbf{E}_S\boldsymbol{\Gamma}_A^2\mathbf{e}$
$\mathbf{E}_S$, $\mathbf{B}_A$	$\mathbf{e}\cdot\mathbf{E}_S\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_S^2\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{B}_A\mathbf{E}_S\mathbf{B}_A^2\mathbf{e}$
$\boldsymbol{\Gamma}_S$, $\mathbf{E}_A$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S\mathbf{E}_A\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S^2\mathbf{E}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_A\boldsymbol{\Gamma}_S\mathbf{E}_A^2\mathbf{e}$
$\boldsymbol{\Gamma}_S$, $\mathbf{B}_A$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_S^2\mathbf{B}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{B}_A\boldsymbol{\Gamma}_S\mathbf{B}_A^2\mathbf{e}$
$\mathbf{B}_S, \boldsymbol{\Gamma}_A$	$\mathbf{e}\cdot\mathbf{B}_S\boldsymbol{\Gamma}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{B}_S^2\boldsymbol{\Gamma}_A\mathbf{e}$	$\mathbf{e}\cdot\boldsymbol{\Gamma}_A\mathbf{B}_S\boldsymbol{\Gamma}_A^2\mathbf{e}$
$\mathbf{B}_S$, $\mathbf{E}_A$	$\mathbf{e}\cdot\mathbf{E}_S\mathbf{E}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_S^2\mathbf{E}_A\mathbf{e}$	$\mathbf{e}\cdot\mathbf{E}_A\mathbf{E}_S\mathbf{E}_A^2\mathbf{e}$

5 Conclusions

We have discussed here the new definition of the material symmetry group $\mathscr{G}_\varkappa$ of the non-linear polar elastic continuum. The group $\mathscr{G}_\varkappa$ consists of an ordered triple of tensors which make the strain energy density invariant under change of the reference placement. Reduced forms of the constitutive equations for the polar-elastic solids are given for several particular cases of material symmetry groups.

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