On refined constitutive equations in the six-field theory of elastic shells

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ABSTRACT: Within the resultant six-field shell theory, the second approximation to the complementary energy density of an isotropic elastic shell undergoing small strains is constructed. In this case, the resultant drilling couples are expressed explicitly by the stress resultants and stress couples as well as by amplitudes of the quadratic and cubic distributions of an intrinsic deviation vector. The refined 2D strain-stress and stress-strain constitutive relations for shells are found, in which the effect of curvatures of the shell midsurface is consistently taken into account.

1 INTRODUCTION

The general non-linear theory of shells, proposed by Reissner (1974) and developed by Libai and Simmonds (1998) and Chróścielewski et al. (2004), is formulated in terms of three translations and three rotations of the base surface as independent field variables. The theory also takes into account two resultant drilling couples with corresponding two work-conjugate drilling bendings. The sixth (drilling) rotation as well as the drilling stress and strain measures become of primary importance in analyses of irregular shells with kinks, branchings and/or intersections, as well as in junctions of shell elements with beams, columns and/or stiffeners.

For an isotropic elastic material undergoing small strains, John (1965) proved that on the cross section of a thin shell the transverse shear stresses are one order smaller than the normal stresses. In the six-field shell model the 2D stress resultants and stress couples are defined by direct through-the-thickness integration of 3D stress distribution applied on the cross section. Thus, the constitutive equations for them should be formulated with a greater accuracy than the ones for the transverse shear resultants and the resultant drilling couples.

Yet, in most numerical finite element analyses of the geometrically non-linear problems of elastic shells the simplest constitutive equations of the linear five-field theory of plates without the drilling couples and without any account of curvatures of the undeformed shell midsurface have been used. In the six-field shell theory the 2D strain measures are defined from the principle of virtual work only on the 2D level without any relation to 3D strain measures. As a result, in the present report the simplest 2D constitutive equations are refined by taking consistently into account the undeformed midsurface curvatures. The refinements are based on the second approximation to the complementary energy density of an isotropic elastic shell undergoing small strains.

2 COMPLEMENTARY ENERGY DENSITY

In non-linear elasticity, from a few forms of the stored energy density several forms of the complementary energy density may be defined applying the Legendre transformation. For our purpose, it is convenient to begin, after Koiter (1976), with the stored energy density per unit undeformed volume $\mathrm{W}=\bar{\mathrm{W}}(\boldsymbol{\varepsilon})$, where $\boldsymbol{\varepsilon}=\mathbf{U}-\mathbf{1}=\boldsymbol{\varepsilon}^T$ is the relative stretch tensor with the right stretch tensor $\mathbf{U}=\mathbf{U}^T$ following from the polar decomposition of the deformation gradient $\mathbf{F}=\mathbf{RU}$. Differentiating the density $\bar{\mathrm{W}}(\boldsymbol{\varepsilon})$ we obtain

$$\frac{\partial\bar{\mathrm{W}}}{\partial\boldsymbol{\varepsilon}}=\mathbf{T}=\frac{1}{2}\left(\mathbf{SU}+\mathbf{US}\right)=\mathrm{T}^{ij}\mathbf{g}_i\otimes\mathbf{g}_j\ , \qquad (1)$$

where $\mathbf{T}=\mathbf{T}^T$ is the Jaumann stress tensor, $\mathbf{S}=\mathbf{S}^T$ is the 2nd Piola-Kirchhoff stress tensor, $\mathbf{g}_i$ are base vectors of the 3D undeformed curvilinear coordinates $\theta^i, i=1,2,3,$ and $\otimes$ is the tensor product.

For an isotropic elastic solid, when $\mathbf{T}$ and $\boldsymbol{\varepsilon}$ are coaxial, Koiter (1976) proved that (1) can be uniquely inverted to the form $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\mathbf{T})$ provided that the rotations $\mathbf{R}$ are at most moderate.

The elastic range of many engineering materials is restricted to small strains such that $\|\boldsymbol{\varepsilon}\|<<1$, and the constitutive equations are governed by the

Hooke law. Since under small strains $\mathbf{T} \simeq \mathbf{S}$, the complementary energy density $\bar{\mathrm{W}}_c(\mathbf{S})$ following by the Legendre transformation of $\bar{\mathrm{W}}(\boldsymbol{\varepsilon})$ becomes

$$\bar{\mathrm{W}}_c(\mathbf{S}) = \mathrm{tr}[\mathbf{S}\boldsymbol{\varepsilon}(\mathbf{S})] - \bar{\mathrm{W}}[\boldsymbol{\varepsilon}(\mathbf{S})] = \frac{1}{2}\mathrm{K}_{ijkl}\mathrm{S}^{ij}\mathrm{S}^{kl} . \qquad (2)$$

For an isotropic elastic material the 3D elastic compliances are

$$\mathrm{K}_{ijkl} = \frac{1}{2E}\Big[(1+\nu)\big(\mathrm{g}_{ik}\mathrm{g}_{jl} + \mathrm{g}_{il}\mathrm{g}_{jk}\big) - 2\nu\mathrm{g}_{ij}\mathrm{g}_{kl}\Big],$$

with E the Young modulus, ν the Poisson ratio, and $\mathrm{g}_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$.

2 SOME SHELL RELATIONS

A shell is a three-dimensional (3D) solid body identified in a reference (undeformed) placement with a thin region B of the physical space. The shell boundary $\partial\mathrm{B}$ consists of three separable parts: the upper M^+ and lower M^- shell faces, and the lateral shell boundary surface $\partial\mathrm{B}^*$. The position vectors $\mathbf{x}$ and $\mathbf{y} = \chi(\mathbf{x})$ of any material particle in the reference and deformed placements, respectively, can conveniently be represented by

$$\mathbf{x} = \boldsymbol{x} + \xi\boldsymbol{n}, \quad \mathbf{y} = \boldsymbol{y}(\boldsymbol{x}) + \boldsymbol{\zeta}(\boldsymbol{x},\xi), \quad \boldsymbol{\zeta}(\boldsymbol{x},0) = \mathbf{0} .$$

Here $\boldsymbol{x}$ and $\boldsymbol{y}$ are position vectors of some shell base surface M and $N = \chi(M)$ in the reference and deformed placements, respectively, ξ is the distance from M along the unit normal vector $\boldsymbol{n}$ orienting M such that $\xi \in [-h^-, h^+]$, $h = h^- + h^+$ is the shell thickness, $\boldsymbol{\zeta}$ is a deviation vector of $\mathbf{y}$ from N, while χ and χ mean the 3D and 2D deformation functions, respectively. Geometry of B is usually described in the normal coordinates (θ^α, ξ), $\alpha = 1,2$, such that the corresponding base vectors of M are given by $\boldsymbol{a}_\alpha = \boldsymbol{x},_\alpha$, $\boldsymbol{n} = (1/2)\varepsilon^{\alpha\beta}\boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta$ leading to the surface metric $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ and curvature $b^\alpha_\beta = -\boldsymbol{a}^\alpha \cdot \boldsymbol{n},_\beta$ tensors.

Within the six-field theory of shells presented in Chróścielewski et al. (2004), the referential internal contact stress resultant $\boldsymbol{n}_\nu = \boldsymbol{n}^\alpha \nu_\alpha$ and stress couple $\boldsymbol{m}_\nu = \boldsymbol{m}^\alpha \nu_\alpha$ vectors, defined at the edge ∂R of an arbitrary part of the deformed base surface $R = \chi(P)$, $P \subset M$, but measured per unit length of the undeformed edge ∂P having the outward unit normal vector $\boldsymbol{\nu} = \nu_\alpha \boldsymbol{a}^\alpha$, are defined by

$$\boldsymbol{n}^\alpha = \int_{-h^-}^{+h^+} \mathrm{S}^{\alpha i}\mathbf{F}\mathbf{g}_i \mu \mathrm{d}\xi, \;\; \boldsymbol{m}^\alpha = \int_{-h^-}^{+h^+} \boldsymbol{\zeta} \times \mathrm{S}^{\alpha i}\mathbf{F}\mathbf{g}_i \mu \mathrm{d}\xi. \quad (3)$$

The unique 2D shell kinematics associated with M consists of the translation vector $\boldsymbol{u}$ and the proper orthogonal (rotation) tensor $\boldsymbol{Q}$, both describing the gross deformation (work-averaged through the shell thickness) of the shell cross section,

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}, \quad \boldsymbol{t}_\alpha = \boldsymbol{Q}\boldsymbol{a}_\alpha , \quad \boldsymbol{t} = \boldsymbol{Q}\boldsymbol{n},$$

where $\boldsymbol{t}_\alpha, \boldsymbol{t}$ are three directors attached to any point of $N = \chi(M)$. As a result, $\boldsymbol{n}^\alpha$ and $\boldsymbol{m}^\alpha$ can naturally be represented in components relative to the rotated base $\boldsymbol{t}_\alpha, \boldsymbol{t}$ by

$$\boldsymbol{n}^\alpha = N^{\alpha\beta}\boldsymbol{t}_\beta + Q^\alpha \boldsymbol{t}, \quad \boldsymbol{m}^\alpha = \boldsymbol{t} \times M^{\alpha\beta}\boldsymbol{t}_\beta + M^\alpha \boldsymbol{t}.$$

The 2D strain $\boldsymbol{\varepsilon}_\alpha$ and bending $\boldsymbol{\kappa}_\alpha$ vectors work-conjugate to the respective stress resultant $\boldsymbol{n}^\alpha$ and stress couple $\boldsymbol{m}^\alpha$ vectors are defined by

$$\boldsymbol{\varepsilon}_\alpha = \boldsymbol{y},_\alpha - \boldsymbol{t}_\alpha = \boldsymbol{u},_\alpha + (\mathbf{1} - \boldsymbol{Q})\boldsymbol{a}_\alpha = E_{\alpha\beta}\boldsymbol{t}^\beta + E_\alpha \boldsymbol{t},$$

$$\boldsymbol{\kappa}_\alpha = \mathrm{ax}\left(\boldsymbol{Q},_\alpha \boldsymbol{Q}^T\right) = \boldsymbol{t} \times K_{\alpha\beta}\boldsymbol{t}^\beta + K_\alpha \boldsymbol{t},$$

where $\mathbf{1}$ is the metric tensor of the 3D space and $\mathrm{ax}(\cdot)$ is the axial vector of the skew tensor $(\cdot)$.

For what follows it is convenient to introduce the referential deviation vector $\mathbf{e}(\boldsymbol{x},\xi)$,

$$\mathbf{e} = \boldsymbol{Q}^T\boldsymbol{\zeta} - \xi\boldsymbol{n} = \mathrm{e}^\rho \mathbf{g}_\rho , \quad \mathbf{e}(\boldsymbol{x},0) = \mathbf{0}. \qquad (4)$$

In shell theory the rotational part of deformation is described by the tensor $\boldsymbol{Q}$. Thus, it is natural to apply here the modified polar decomposition of $\mathbf{F}$ in the form, see Pietraszkiewicz et al. (2006),

$$\mathbf{F}(\boldsymbol{x},\xi) = \boldsymbol{Q}(\boldsymbol{x})[\mathbf{1} + \boldsymbol{\Theta}(\boldsymbol{x},\xi)] , \quad \boldsymbol{\Theta} \neq \boldsymbol{\Theta}^T .$$

If the largest stretch η in the shell space is assumed to be small, then $\|\boldsymbol{\Theta}\| << 1$. Let us also assume the vector $\mathbf{e}$ to be one order smaller as compared with h, so that $(|\mathbf{e}|/h)^2 << 1$. Then consistently omitting the corresponding small terms with respect to the unity, the shell stress resultants and stress couples follow now from approximations

$$N^{\alpha\beta} \simeq \int_-^+ \mathrm{S}^{\alpha\psi}\mu^\beta_\psi \mu \mathrm{d}\xi , \quad Q^\alpha \simeq \int_-^+ \mathrm{S}^{\alpha 3}\mu \mathrm{d}\xi , \qquad (5)$$

$$M^{\alpha\beta} \simeq \int_-^+ \mathrm{S}^{\alpha\psi}\mu^\beta_\psi \mu\xi \mathrm{d}\xi , \quad \int_-^+ \equiv \int_{-h^-}^{+h^+} ,$$

$$M^\alpha \simeq \int_-^+ (\mathrm{S}^{\alpha\psi}\mu^\beta_\psi \mu)\varepsilon_{\gamma\beta}\mu^\gamma_\rho \mathrm{e}^\rho \mathrm{d}\xi , \qquad (6)$$

where $\mu^\beta_\psi = \delta^\beta_\psi - \xi b^\beta_\psi$ and $\mu = |\mu^\beta_\psi|$.

3 SECOND APPROXIMATION TO COMPLEMENTARY ENERGY DENSITY

Taking into account symmetries of K_{ijkl} and S^{ij} , the quadratic expression (2) can be written as the sum of four terms each representing a part of 3D complementary energy density calculated from the stresses $\mathrm{S}^{\varphi\psi}$, $\mathrm{S}^{\varphi 3} = \mathrm{S}^{3\varphi}$ and S^{33}. However, the stress component S^{33} acts on the shell surfaces $\xi = \mathrm{const}$ parallel to the base surface M . Although S^{33} contributes to the 3D complementary energy density, it does not enter the resultant 2D equilibrium equations on M and does not contribute to the effective part $\bar{\mathrm{W}}^{eff}_c$ of $\bar{\mathrm{W}}_c$ associated with the resultants (5) and (6). Thus,

$$\bar{\mathrm{W}}_c^{eff} = \frac{1}{2\mu^2}\Big[A_{\alpha\beta\lambda\mu}\mu_\varphi^\alpha\left(\mu S^{\varphi\psi}\mu_\psi^\beta\right)\mu_\theta^\lambda\left(\mu S^{\theta\sigma}\mu_\sigma^\mu\right) + 4A_{\alpha3\lambda3}\mu_\varphi^\alpha\left(\mu S^{\varphi3}\right)\mu_\theta^\lambda\left(\mu S^{\theta3}\right)\Big], \tag{7}$$

where $A_{ijkl} = K_{ijkl}|_{\xi=0}$.

Let M be the middle surface of the shell in the undeformed placement, so that $h^- = h^+ = h/2$. Assume also that there are no surface forces applied at the upper and lower shell faces $M^\pm$, and no body forces applied in the internal shell space. Then to within the bulk terms distribution of the stresses in the shell space can, in fact, be approximately represented by the resultant terms according to

$$\mu S^{\alpha\psi}\mu_\psi^\beta \simeq \frac{1}{h}N^{\alpha\beta} + \frac{12}{h^3}M^{\alpha\beta}\xi, \qquad \mu S^{\alpha3} \simeq \frac{1}{h}Q^\alpha f(\xi), \quad f(\xi) = \frac{3}{2}\left(1 - \frac{4\xi^2}{h^2}\right). \tag{8}$$

For thin isotropic elastic shells undergoing small strains John (1965) obtained concrete quantitative error estimates for stresses and their derivatives in the case of vanishing surface and body forces. With additional physically motivated estimates proposed by Koiter (1980), we can estimate orders of some fields appearing in shell theory as follows:

$$A_{\alpha\beta\lambda\mu} \sim \frac{1}{E},\ A_{\alpha3\lambda3} \sim \frac{\nu}{E},\ S^{\varphi\psi} \sim E\eta,\ S^{\varphi3} \sim E\eta\theta, \tag{9}$$

$$\theta = \max_{x\subset M}\left(\frac{h}{L}, \frac{h}{d}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\right),\ \theta^2 << 1,\ a_{\alpha\beta} \sim 1,\ b_\alpha^\beta \sim \frac{\theta^2}{h}$$

where $\sim$ means "of the order of", L is the smallest characteristic length of geometric, extensional and bending deformation patterns on M, respectively, d is the distance of internal shell points to the shell boundary, and θ is the common small parameter.

Assuming that the stresses $S^{\varphi\psi}$ entering definitions (5) of $N^{\alpha\beta}$ and (6) of $M^{\alpha\beta}$ are of the same order, from (9) we obtain the estimates

$$N^{\alpha\beta} \sim Eh\eta, \quad M^{\alpha\beta} \sim Eh^2\eta, \quad Q^\alpha \sim Eh\eta\theta. \tag{10}$$

Let $e^\rho(\xi)$ introduced in (4) be approximated by the sum of quadratic and cubic polynomials of ξ,

$$e^\rho(\xi) = q^\rho k(\xi) + c^\rho g(\xi), \qquad k(\xi) = \frac{4\xi^2}{h^2}, \quad g(\xi) = \frac{5}{2}\xi\left(1 - \frac{4\xi^2}{h^2}\right). \tag{11}$$

The assumption $(|\mathbf{e}|/h)^2 << 1$ above (5) means that in terms of θ we have assigned orders of the amplitudes in $(11)_1$ to be $q^\rho \sim hc^\rho \sim h\theta$. Unfortunately, we are not aware of any estimation for q^ρ and c^ρ available in the literature for the geometrically non-linear theory of elastic shells.

With the estimates (9) and (10) it follows that within the relative accuracy θ^2 we can approximate M^α only by two principal terms,

$$M^\alpha = \varepsilon_{\rho\beta}\frac{1}{3}N^{\alpha\beta}q^\rho + \varepsilon_{\rho\beta}M^{\alpha\beta}c^\rho + O(Eh^2\eta\theta^3), \tag{12}$$

where $O(Eh^2\eta\theta^3)$ means all the remaining terms $\sim Eh^2\eta\theta^3$. The relation indicates that M^α can be established if $N^{\alpha\beta}, M^{\alpha\beta}$ and q^ρ, c^ρ are known, and that contribution of the drilling couples to the stress distribution across the shell thickness is of smaller order than contribution of $N^{\alpha\beta}$ and $M^{\alpha\beta}$.

The 2D effective complementary energy density Σ^{eff} of the shell can now be obtained by direct through-the-thickness integration of the corresponding 3D density,

$$\Sigma_c^{eff} = \int_-^+ \mu\bar{\mathrm{W}}_c^{eff}\, d\xi. \tag{13}$$

Taking into account that $1/\mu = 1 + \xi b_\kappa^\kappa + \ldots$ and introducing (8) into (13) we can express the integrand of (13) by the infinite series of the resultant stress measures, curvatures of M, material parameters, as well as polynomials of h^n and ξ^n, $n = 0,1,2,\ldots$. With the relations (9) and (10) as well as through-the-thickness integration we are able to estimate the order of any term appearing in the infinite series (13). This leads to the following consistent expression for Σ_c^{eff} involving only two principal terms $\sim Eh\eta^2$ and four secondary terms $\sim Eh\eta^2\theta^2$:

$$\Sigma_c^{eff} = \frac{1}{h}A_{\alpha\beta\lambda\mu}\left\{\frac{1}{2}\left(N^{\alpha\beta}N^{\lambda\mu} + \frac{12}{h^2}M^{\alpha\beta}M^{\lambda\mu}\right) + b_\kappa^\kappa N^{\alpha\beta}M^{\lambda\mu} - \left(N^{\alpha\beta}b_\rho^\lambda M^{\rho\mu} + M^{\alpha\beta}b_\rho^\lambda N^{\rho\mu}\right)\right\} + 2\frac{1}{h}A_{\alpha3\lambda3}\frac{1}{\alpha_s}Q^\alpha Q^\lambda + O(Eh\eta^2\theta^3), \tag{14}$$

where the shear correcting factor $\alpha_s = 5/6$.

The refined constitutive equations for 2D strain measures follow now from differentiation of (14) with regard to appropriate resultant stress measures:

$$E_{\alpha\beta} = \frac{\partial\Sigma_c^{eff}}{\partial N^{\alpha\beta}} = \frac{1}{h}A_{\alpha\beta\lambda\mu}\left(N^{\lambda\mu} - b_\rho^\lambda M^{\rho\mu} + b_\kappa^\kappa M^{\lambda\mu}\right) - \frac{1}{h}b_\alpha^\kappa A_{\kappa\beta\lambda\mu}M^{\lambda\mu} + O(\eta\theta^3), \tag{15}$$

$$K_{\alpha\beta} = \frac{\partial\Sigma_c^{eff}}{\partial M^{\alpha\beta}} = \frac{1}{h}A_{\alpha\beta\lambda\mu}\left(\frac{12}{h^2}M^{\lambda\mu} - b_\rho^\lambda N^{\rho\mu} + b_\kappa^\kappa N^{\lambda\mu}\right) - \frac{1}{h}b_\alpha^\kappa A_{\kappa\beta\lambda\mu}N^{\lambda\mu} + O\left(\frac{\eta}{h}\theta^3\right), \tag{16}$$

$$E_\alpha = \frac{\partial\Sigma^{eff}}{\partial Q^\alpha} = \frac{4}{\alpha_s h}A_{\alpha3\lambda3}Q^\lambda + O\left(\eta\theta^3\right). \tag{17}$$

The relations (17) can easily be solved for Q^α,

$$Q^{\alpha} = \alpha_s \frac{Eh}{2(1+\nu)} a^{\alpha\lambda} E_{\lambda} + O(Eh\eta\theta^3) .$$

The relations (15) and (16) constitute the set of eight linear inhomogeneous algebraic equations for eight non-symmetric components $N^{\alpha\beta}$ and $M^{\alpha\beta}$. They can always be inverted numerically in any particular coordinates θ^{α} provided that determinant of 8×8 matrix of their coefficients does not vanish.

For example, if θ^{α} are the arc lengths of orthogonal lines of principal curvatures of M , then

$$a_{11} = a_{22} = 1 ,\ a_{12} = 0 ,\ \sqrt{a} = 1 ,\ A_{1111} = A_{2222} = \frac{1}{E} ,$$

$$b_1^1 = -\frac{1}{R_1} ,\ b_2^2 = -\frac{1}{R_2} ,\ b_2^1 = b_1^2 = 0 ,\ A_{1112} = A_{2212} = 0 ,$$

$$A_{1122} = -\frac{\nu}{E} ,\ A_{1212} = A_{1313} = A_{2323} = \frac{1+\nu}{2E} , \qquad (18)$$

where R_1 and R_2 are principal radii of curvatures of M . With (18) the eight algebraic equations (15) and (16) can then be written as two separate sets of four algebraic equations, which in matrix form read

$$\mathbf{D}_1 = \mathbf{A}\mathbf{S}_1 , \quad \mathbf{D}_2 = \mathbf{B}\mathbf{S}_2 , \qquad (19)$$

$$\mathbf{D}_1 = \left[E_{11}, E_{22}, K_{11}, K_{22}\right]^T ,\ \mathbf{S}_1 = \left[N^{11}, N^{22}, M^{11}, M^{22}\right]^T ,$$

$$\mathbf{D}_2 = \left[E_{12}, E_{21}, K_{12}, K_{21}\right]^T ,\ \mathbf{S}_2 = \left[N^{12}, N^{21}, M^{12}, M^{21}\right]^T .$$

It can be proved that the matrix $\mathbf{A}$ in (19) is non-singular for any geometry of M and the matrix $\mathbf{B}$ in (19) is non-singular provided that $R_1 \neq R_2$. Thus, by inverting (19) with the second-order accuracy we can find the consistently refined constitutive equations, for example

$$N^{11} = C\left(E_{11} + \nu E_{22}\right) - D\left(\frac{1}{R_1} - \frac{1}{R_2}\right) K_{11} + O(Eh\eta\theta^3)$$

$$N^{12} = \frac{1}{2} C(1-\nu)\left(E_{12} + E_{21}\right) - D(1-\nu)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) K_{12}$$

$$+ O(Eh\eta\theta^3) , \quad C = \frac{Eh}{1-\nu^2} , \quad D = \frac{Eh^3}{12\left(1-\nu^2\right)} .$$

The constitutive equations for M^{α} follow now from (12), where within the indicated error only the first main terms in the expressions (15) for $E_{\alpha\beta}$ and (16) for $K_{\alpha\beta}$ should be taken into account. Then inverting such first-approximation expressions for $N^{\alpha\beta}$ and $M^{\alpha\beta}$ and taking account of symmetries of $A_{\alpha\beta\lambda\mu}$, we obtain

$$N^{(\alpha\beta)} = hH^{\alpha\beta\lambda\mu} E_{(\lambda\mu)} + O(Eh\eta\theta^2) ,$$
$$M^{(\alpha\beta)} = \frac{h^3}{12} H^{\alpha\beta\lambda\mu} K_{(\lambda\mu)} + O(Eh^2\eta\theta^2) , \qquad (20)$$

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta}a^{\lambda\mu}\right) .$$

Introducing (27) into (19) yields the following constitutive equations for M^{α} :

$$M^{\alpha} = \varepsilon_{\rho\beta} H^{\alpha\beta\lambda\mu}\left(\frac{1}{3} hE_{(\lambda\mu)} q^{\beta} + \frac{h^2}{12} K_{(\lambda\mu)} c^{\beta}\right)$$
$$+ O(Eh^2\eta\theta^3) , \qquad (21)$$

where $E_{(\lambda\mu)}$ means the symmetric part of $E_{\lambda\mu}$.

4 CONCLUSIONS

We have presented the strain-stress and stress-strain constitutive equations for the geometrically non-linear theory of an isotropic elastic shell which are refined by the undeformed midsurface curvature. They are based on the second approximation to the shell complementary energy density (14). In particular, the constitutive equations for the resultant drilling couples have been proved to be expressible explicitly by the 2D strain measures and amplitudes of quadratic and cubic distributions across the shell thickness of the intrinsic deviation vector.

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REFERENCES

Chróścielewski, J., Makowski, J. & Pietraszkiewicz, W. 2004. *Statics and Dynamics of Multi-Shells: Nonlinear Theory and Finite Element Method* (in Polish). Warsaw: IFTR PASci Press.

John, F. 1965. Estimates for the derivatives of the stresses in a thin shell and interior shell equations. *Communications in Pure and Applied Mathematics* 18(1): 235-267.

Koiter, W.T. 1980. The intrinsic equations of shell theory with some applications. In S. Nemat-Nasser (ed.), *Mechanics Today* 5: 139-154. Oxford: Pergamon Press.

Koiter, W.T. 1976. On the complementary energy theorem in non-linear elasticity theory. In: G. Fichera (ed.), *Trends in Applications of Pure Mathematics to Mechanics*: 207-232. London: Pitman.

Libai, A. &. Simmonds J.G. 1998. *The Nonlinear Theory of Elastic Shells,* 2nd ed.` Cambridge, UK: Cambridge Univ. Press.

Pietraszkiewicz, W., Chróścielewski, J., &. Makowski J. 2006 On dynamically and kinematically exact theory of shells. In W. Pietraszkiewicz & C. Szymczak (eds.), *Shell Structures: Theory and Applications*: 163-167. Leiden: Taylor & Francis/Balkema.

Reissner, E. 1974. Linear and nonlinear theory of shells. In Y.C. Fung & E.E. Sechler (eds.), *Thin Shell Structures*: 29-44. Englewood Cliffs, NJ: Prentice-Hall.