Singular curves in the resultant thermomechanics of shells

Wojciech Pietraszkiewicz[a], Violetta Konopińska[b]
[a]*Institute of Fluid-Flow Machinery, PASci, ul. Gen. J. Fiszera 14, 80-231 Gdańsk, Poland*
[b]*Gdańsk University of Technology, Department of Structural Mechanics and Bridge Structures, ul. G. Narutowicza 11/12, 80-952 Gdańsk, Poland*

Dedicated to Professor L.M. Zubov on the occasion of His 70th birthday

ABSTRACT Some geometric and kinematic relations associated with the curve moving on the shell base surface are discussed. The extended surface transport relation and the extended surface divergence theorems are proposed for the piecewise smooth tensor fields acting on the regular and piecewise regular surfaces. The recently formulated resultant, two-dimensionally exact, thermodynamic shell relations - the balances of mass, linear and angular momenta, and energy as well as the entropy inequality – are completed by the corresponding jump conditions at the moving (non-material) and stationary (material) singular surface curves.

Keywords: singular surface curve, jump condition, irregular shell, shell thermomechanics

1 Introduction

Moving singular surfaces are used in three-dimensional (3D) continuum thermomechanics to properly model such phenomena as wave propagation, phase transition, strain localization, fracture, etc. We refer to known books for example by Truesdell and Toupin (1960), Truesdell and Noll (1965), Kosiński (1986), Šilhavý (1997), Gurtin (2000), Bhattacharya (2003), Abeyaratne and Knowles (2006), Berezovski et al. (2008), Gurtin et al. (2010) and the extensive literature given there.

If a thin body - a shell - undergoes a phase transition or a shock wave, for example, there is a thin, small, time-dependent region of material points where physical quantities change rapidly in space and time. For description of such processes within phenomenological shell models, a Gibbs type singular, moving, surface curves can be used across which jump discontinuities of some physical surface fields occur. In shell thermomechanics some one-dimensional (1D) jump conditions were formulated and applied only in a few papers using different shell models. In particular, within a version of the six-field shell thermomechanics the description of singular surface curves and jump conditions across them were proposed in the report by Makowski and Pietraszkiewicz (2002). Some versions of the jump conditions were used by Eremeyev and Zubov (2008) and Eremeyev and Pietraszkiewicz (2009, 2010) to model phase transition phenomena in shells within alternative models of shell thermomechanics.

The refined, resultant, two-dimensional (2D) balance laws of mass, linear and angular momenta, and energy as well as entropy inequality of the non-linear theory of shells were formulated by Pietraszkiewicz (2011) by direct through-the-thickness integration of corresponding 3D laws of rational thermomechanics proposed by Truesdell and Toupin (1960). Since the resultant 2D stress power could not be expressed exactly through the 2D stress and strain measures alone, an additional surface stress power field w_ν, called an interstitial working after Dunn and Serrin (1985), was added to the resultant 2D balance of energy. In this version of shell thermomechanics only one mean referential temperature $\theta > 0$ was associated with the shell base surface. This was induced by only one scalar balance of energy available in 3D rational thermomechanics. The non-uniform distribution of temperature across the thickness was accounted for in two additional resultant surface fields - the extra heat supply s and the extra entropy supply vector $\boldsymbol{s}$ - for which 2D constitutive equations are required. As a result, the 2D 1st and 2nd laws of the resultant shell thermomechanics could be regarded as exact resultant implications of corresponding 3D laws of rational thermomechanics.

In the undeformed (reference) and deformed (current) placements the shell base surface was assumed in Pietraszkiewicz (2011) to be the regular geometric surface with piecewise smooth boundary. Also all the surface fields appearing in the resultant 2D laws were assumed to be smooth everywhere, so that the classical forms of the surface divergence theorem were applicable. In the present paper those simplifying assumptions are removed, which allows us to complete the resultant 2D laws of shell thermomechanics of Pietraszkiewicz (2011) with corresponding 1D jump (called also continuity) conditions at stationary and/or moving singular surface curves.

The jump conditions at moving singular surface curves are necessary to complete the initial-boundary value problems modeling phase transition and/or wave propagation phenomena within the resultant thermomechanic shell theory. Results available on phase transition phenomena in shells obtained within some simpler 2D shell models have been reviewed by Eremeyev and Pietraszkiewicz (2004, 2010).

The jump conditions at stationary (material) singular surface curves are necessary to model geometric and material irregularities such as folds, branchings, intersections, stepwise thickness changes, abrupt changes of material parameters, shell-to-beam and shell-to-column connections, technological junctions, etc. Mechanical problems of such geometric, kinematic and/or mechanical irregularities in shells were discussed using various analytical and numerical shell models for example by Makowski and Stumpf (1994), Bernadou and Cubier (1998), Chróścielewski et al. (1996, 1997, 2002, 2004, 2011), Makowski et al. (1998, 1999), Pietraszkiewicz (2001), Teng (2004), Konopińska and Pietraszkiewicz (2007), and Pietraszkiewicz and Konopińska (2011), where references to other results are given.

This paper is organized as follows. After recalling appropriate definitions of the surface gradient and divergence operators, in section 3 we discuss a surface curve moving on the reference shell base surface. This allows one to extend the surface transport relation for any piecewise smooth tensor field acting on the surface as well as to formulate the surface divergence theorem in the presence of the singular surface curve. Then in section 4 we derive 2D analogues of the geometric and kinematic compatibility conditions on the surface with a singular curve. The referential 1D jump conditions at the moving (non-material) singular curve, which are associated with the 2D balances of mass, linear and angular momenta, and energy as well as with the entropy inequality, are derived in section 5. Finally, section 6 provides the referential 1D jump conditions associated with the above five basic laws of resultant shell thermomechanics at stationary singular surface curves of the irregular shell base surface.

2 Preliminaries

In the undeformed (reference) placement the regular shell is usually represented by the regular smooth base surface M, which in the deformed (current) placement is also assumed to be the regular smooth surface $M(t)=\chi(M,t)$, where χ is a deformation function and t is time. By $x \in M$ and $y=\chi(x,t)\in M(t)$ we denote corresponding placements of a material particle of the base surface in the 3D physical space $\mathcal{E}$ with E as its translation vector space. Then $\boldsymbol{x}=x-\mathrm{o}\in E$ and $\boldsymbol{y}=y-\mathrm{o}\in E$ are the respective position vectors of x and y in an inertial frame $(\mathrm{o},\mathbf{e}_i)$, where $\mathrm{o}\in\mathcal{E}$ is an origin and $\mathbf{e}_i \in E$, $i=1,2,3,$ are orthonormal vectors. The base surface M is oriented by a choice of unit normal vector $\boldsymbol{n}(x)$. The space of all vectors perpendicular to $\boldsymbol{n}(x)$ is then the tangent space T_xM at $x\in M$, and a vector field $\boldsymbol{t}$ on M is tangential if $\boldsymbol{t}(\mathrm{x})\in T_xM$ at every $x\in M$. Given a regular smooth part $\Pi\subset M$ with a piecewise smooth boundary $\partial\Pi$, the outward unit normal $\boldsymbol{\nu}$ at regular $x \in \partial\Pi$ is directed outward of $\partial\Pi$ and tangent to M.

Let $\varphi(x) \in R$, $\boldsymbol{a}(x) \in E$, and $\boldsymbol{T}(x) \in E \otimes E$ be smooth scalar-valued, vector-valued, and 2^{nd}-order tensor-valued fields on M, respectively. Then the surface gradient operator $Grad$ applied to the fields $\varphi, \boldsymbol{a}, \boldsymbol{T}$ leads to $Grad\,\varphi(x) \in T_x M$, a tangential vector field, $Grad\,\boldsymbol{a}(x) \in E \otimes T_x M$, a mixed 2^{nd}-order tensor field, and $Grad\,\boldsymbol{T}(x) \in E \otimes E \otimes T_x M$, a mixed 3^{rd}-order tensor field. These gradient fields are defined as follows: Given any smooth curve C on M described by the position vector $\boldsymbol{x} = \boldsymbol{x}(s)$, where s is the arc length along C, we have

$$\begin{aligned} &\{\varphi(x)\}' = \{Grad\,\varphi(x)\} \cdot \boldsymbol{x}', \quad \{\boldsymbol{a}(x)\}' = \{Grad\,\boldsymbol{a}(x)\}\,\boldsymbol{x}', \\ &\{\boldsymbol{T}(x)\}' = \{Grad\,\boldsymbol{T}(x)\}\,\boldsymbol{x}', \quad (.)' = \frac{d}{ds}, \quad \boldsymbol{x}' = \boldsymbol{\tau}_C, \end{aligned} \tag{1}$$

where $\boldsymbol{\tau}_C \in T_x M$ is the unit tangent vector of C. By definition,

$$(Grad\,\varphi) \cdot \boldsymbol{n} = 0, \quad (Grad\,\boldsymbol{a})\,\boldsymbol{n} = \boldsymbol{0}, \quad (Grad\,\boldsymbol{T})\,\boldsymbol{n} = \boldsymbol{0}. \tag{2}$$

The surface divergence Div of a vector $\boldsymbol{a}(x) \in E$ and a mixed 2^{nd}-order tensor $\boldsymbol{S}(x) \in E \otimes T_x M$ fields on M are defined by

$$\begin{aligned} &Div\,\boldsymbol{a}(x) = \mathrm{tr}\{\boldsymbol{P}\,Grad\,\boldsymbol{a}(x)\}, \\ &\{Div\,\boldsymbol{S}(x)\} \cdot \boldsymbol{c} = \mathrm{tr}\left\{\boldsymbol{P}Grad\left(\boldsymbol{S}^T(x)\boldsymbol{c}\right)\right\} = Div\left\{\boldsymbol{S}^T(x)\boldsymbol{c}\right\}, \quad \text{for any } \boldsymbol{c} \in E, \end{aligned} \tag{3}$$

where $\boldsymbol{P} = \boldsymbol{1} - \boldsymbol{n} \otimes \boldsymbol{n} \in T_x M \otimes E$ is the perpendicular projection onto M and $\boldsymbol{1} \in E \otimes E$ is the unit (metric) tensor of 3D space.

With definitions (1) and (3) the surface divergence theorem valid on any regular, smooth parts Π of M are given by Gurtin and Murdoch (1974), eq. $(2.22)_1$,

$$\int_{\partial\Pi} \boldsymbol{a} \otimes \boldsymbol{S}\boldsymbol{\nu}\,ds = \iint_{\Pi} \left\{\boldsymbol{a} \otimes Div\,\boldsymbol{S} + (Grad\,\boldsymbol{a})\,\boldsymbol{S}^T\right\} da \tag{4}$$

for any smooth (of class C^1) vector $\boldsymbol{a}$ and mixed 2^{nd}-order tensor $\boldsymbol{S}$ fields at $x \in \Pi$. Particular cases of (4) are

$$\begin{aligned} &\int_{\partial\Pi} \boldsymbol{a} \cdot \boldsymbol{\nu}\,ds = \iint_{\Pi} \left(Div\,\boldsymbol{a} + 2Ha_n\right) da, \quad \int_{\partial\Pi} \boldsymbol{S}\boldsymbol{\nu}\,ds = \iint_{\Pi} Div\,\boldsymbol{S}\,da, \\ &\int_{\partial\Pi} \boldsymbol{a} \times \boldsymbol{S}\boldsymbol{\nu}\,ds = \iint_{\Pi} \left\{\boldsymbol{a} \times (Div\,\boldsymbol{S}) + \mathrm{ax}\left(\boldsymbol{S}(Grad\,\boldsymbol{a})^T - (Grad\,\boldsymbol{a})\boldsymbol{S}^T\right)\right\} da, \end{aligned} \tag{5}$$

where $a_n = \boldsymbol{n} \cdot \boldsymbol{a}$ and $H = -(1/2)\,\mathrm{tr}\left(\boldsymbol{P}Grad\boldsymbol{n}\right)$ is the mean curvature in $x \in \Pi$.

It is to be noticed that alternative surface gradient $\nabla\boldsymbol{a}$ and divergence $\nabla \cdot \boldsymbol{T}$ fields can be defined on M with the help of a nabla operator ∇ such that

$$\nabla \boldsymbol{a} = \left(Grad\, \boldsymbol{a}\right)^T , \quad \boldsymbol{c} \cdot \left(\nabla \cdot \boldsymbol{T}\right) = \nabla \cdot \left(\boldsymbol{T}\boldsymbol{c}\right) \quad \text{for any } \boldsymbol{c} \in E . \tag{6}$$

With these alternative definitions (6) the surface divergence theorems (4) and (5) take some alternative forms. In the present paper we shall not use these alternative conventions.

3 Non-material curve moving on the regular reference base surface

The global, resultant, balance laws and entropy inequality of shell thermomechanics formulated in Pietraszkiewicz (2011) each involves the material time derivative of a surface integral. In the absence of singular curves, the standard transport theorem on any fixed part of M allows one to change the order of surface integration and material time differentiation. But for the reference shell base surface containing a moving singular curve the transport relation as well as the divergence theorems (5) have to be carefully extended to take into account the effect of the moving discontinuity.

A surface curve moving on the reference base surface M over a time interval $I = [t_0, t_1]$, $t_0 < t_1$, is a one–parametric family $\{C(t)\}$ of piecewise smooth surface curves $\boldsymbol{x}_C = \boldsymbol{x}_C(s,t)$ parameterised along each $C(t)$ by the length coordinate s and oriented consistently with the orientation of M. With each regular point $x_C \in C(t)$ we can associate the triad of orthonormal vectors: the tangent $\boldsymbol{\tau}_C = \partial \boldsymbol{x}_C / \partial s$, the normal $\boldsymbol{n}_C = \boldsymbol{n}$, and the exterior normal $\boldsymbol{\nu}_C = \boldsymbol{\tau}_C \times \boldsymbol{n}$. Velocity of $C(t)$ relative to M is the tangential vector field $\boldsymbol{\upsilon}(s,t) = \partial \boldsymbol{x}_C / \partial t \in T_x M$. Its exterior normal component $V = \boldsymbol{\upsilon} \cdot \boldsymbol{\nu}_C$ measures the speed with which the curve $C(t)$ transverses the surface M. The tangential component $T = \boldsymbol{\upsilon} \cdot \boldsymbol{\tau}_C$ is a measure of intrinsic velocity of $C(t)$ within itself.

The normal velocity $\boldsymbol{\upsilon}_\nu = V \boldsymbol{\nu}_C$ determines the normal trajectory of the curve $C(t)$: a surface curve that is tangent to $\boldsymbol{\upsilon}_\nu$ at any instant t. The normal trajectory begins at a certain point of the initial curve $C(t_0)$, and different points of $C(t_0)$ determine different normal trajectories.

Let a smooth time-dependent field $\boldsymbol{\Phi}(x,t)$, with $\boldsymbol{\Phi}$ belonging to any finite-dimensional vector space $R, T_x M, E, E \otimes T_x M, E \otimes E$ etc., is defined only on $M \setminus C(t)$, but the value of $\boldsymbol{\Phi}(x,t)$ need not be defined on $C(t)$. Let $\Pi \subset M$ be an arbitrarily fixed, regular region of M containing a portion of $C(t)$ in its interior, see Fig. 1. For the field $\boldsymbol{\Phi}(x,t)$ on $\Pi \setminus C(t)$

we suppose that at each instant t one-sided finite limits of $\boldsymbol{\Phi}(x,t)$ at $C(t)$ exist. We write $\boldsymbol{\Phi}^-$ for the finite limit of $\boldsymbol{\Phi}$ as C is approached from Π^- and $\boldsymbol{\Phi}^+$ for the finite limit of $\boldsymbol{\Phi}$ as C is approached from Π^+, that is for any regular x_C of $C(t)$ and $\lambda > 0$,

$$\boldsymbol{\Phi}^{\mp}(x_C,t) = \lim_{\lambda \to 0} \boldsymbol{\Phi}\left(x \mp \lambda \boldsymbol{\nu}_C(x_C,t), t\right). \tag{7}$$

Then $[\boldsymbol{\Phi}](s,t)$ and $<\boldsymbol{\Phi}>(s,t)$ denote the jump and the mean value of $\boldsymbol{\Phi}$ at $x_C \in C(t)$ given by

$$[\boldsymbol{\Phi}] = \boldsymbol{\Phi}^+ - \boldsymbol{\Phi}^- , \quad <\boldsymbol{\Phi}> = \frac{1}{2}\left(\boldsymbol{\Phi}^+ + \boldsymbol{\Phi}^-\right). \tag{8}$$

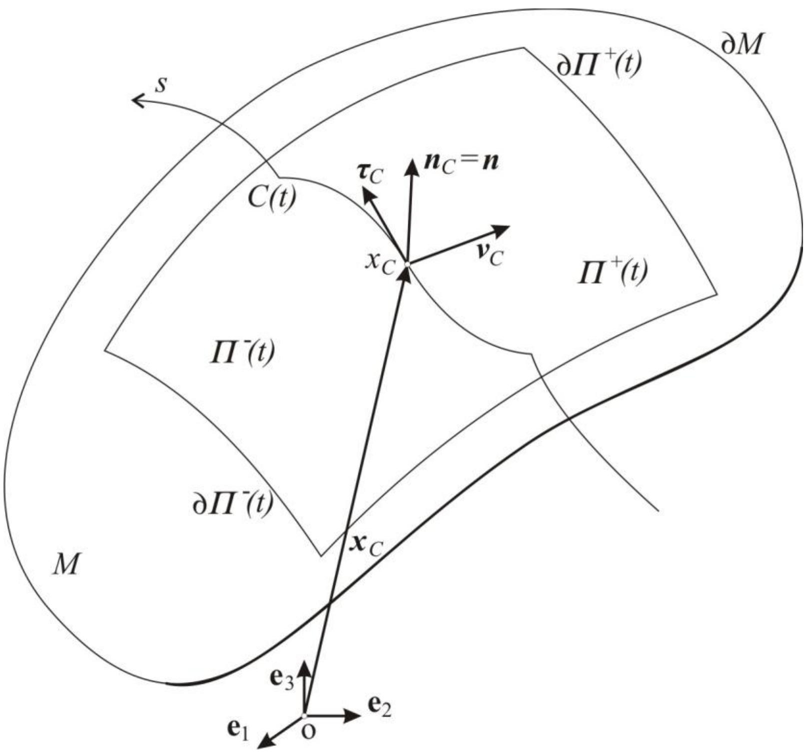

Figure 1. The regular surface M with moving surface curve $C(t)$

If $[\boldsymbol{\Phi}]$ does not vanish identically at all regular points of $C(t)$, the curve $C(t)$ is said to be *singular of the* 0^{th} *order* with respect to $\boldsymbol{\Phi}$ at time t. If $\boldsymbol{\Phi}(x,t)$ is continuous across $C(t)$ then $[\boldsymbol{\Phi}] = \mathbf{0}$ and $<\boldsymbol{\Phi}> = \boldsymbol{\Phi}$, because by continuity $\boldsymbol{\Phi}^- = \boldsymbol{\Phi}^+$.

When $\boldsymbol{\Phi}(x,t)$ is piecewise smooth on Π with $C(t)$ then $Grad\boldsymbol{\Phi}$ is piecewise continuous and $\left(Grad\,\boldsymbol{\Phi}\right)^{\mp}(x_C,t)$ exist at all regular points of $C(t)$ with $[Grad\,\boldsymbol{\Phi}] = \left(Grad\,\boldsymbol{\Phi}\right)^+ - \left(Grad\,\boldsymbol{\Phi}\right)^-$. If the surface gradient of $\boldsymbol{\Phi}$ is continuous and $[\boldsymbol{\Phi}]$ does not vanish identically at all regular points of $C(t)$, then $C(t)$ is *singular of the* 1st *order* with respect to $\boldsymbol{\Phi}$ at time t.

In general, if $\boldsymbol{\Phi}(x,t)$ is *n*-times differentiable on Π with $C(t)$ and its surface gradients of *n*-th order have the finite one-sided limits $\left(Grad^n\,\boldsymbol{\Phi}\right)^{\mp}(x_C,t)$ at all regular points of $C(t)$, then $[Grad^n\,\boldsymbol{\Phi}]=\left(Grad^n\,\boldsymbol{\Phi}\right)^{+}-\left(Grad^n\,\boldsymbol{\Phi}\right)^{-}$. If all $(n-1)$th surface gradients are continuous at $C(t)$ and only $[Grad^n\,\boldsymbol{\Phi}]$ does not vanish identically at all regular points of $C(t)$, the curve $C(t)$ may be called *singular of the n*th *order* with respect to $\boldsymbol{\Phi}$ at time t.

Let $\Pi\subset M$ be an arbitrary fixed, closed, regular region of M containing a part of $C(t)$ in its interior. The curve $C(t)$ separates the region $\Pi\subset M$ into two disjoint complementary subregions $\Pi^{-}(t)$ and $\Pi^{+}(t)$ such that $\Pi^{-}(t)\cup\Pi^{+}(t)=\Pi$ which boundaries $\partial\Pi^{-}(t)$ and $\partial\Pi^{+}(t)$ in the neighbourhood of $C(t)$ are moving, see Fig. 1, so that $\Pi^{-}(t)\cap\Pi^{+}(t)=C(t)$. At each regular point of $C(t)$ the exterior normal vector $\boldsymbol{\nu}^{-}$ of $\partial\Pi^{-}(t)$ coincides with the unit vector $\boldsymbol{\nu}_C$ of $C(t)$. Thus, the exterior normal velocity $\boldsymbol{\upsilon}\cdot\boldsymbol{\nu}^{-}=\boldsymbol{\upsilon}\cdot\boldsymbol{\nu}_C$ of $\partial\Pi^{-}(t)$ is equal to V at $C(t)$ and vanishes elsewhere on $\partial\Pi^{-}(t)$. Likewise, at each regular point of $C(t)$ the vector $\boldsymbol{\nu}^{+}$ of $\partial\Pi^{+}(t)$ coincides with $-\boldsymbol{\nu}_C$ of $C(t)$. Hence, the exterior normal velocity $\boldsymbol{\upsilon}\cdot\boldsymbol{\nu}^{+}=-\boldsymbol{\upsilon}\cdot\boldsymbol{\nu}_C$ of $\partial\Pi^{+}(t)$ becomes $-V$ at $C(t)$ and vanishes elsewhere on $\partial\Pi^{+}(t)$. For the field $\boldsymbol{\Phi}(x,t)$ smooth on the subregions $\Pi^{-}(t)$ and $\Pi^{+}(t)$, by the Reynolds transport theorem for smoothly evolving subregions $\Pi^{\mp}(t)$ with moving boundaries $\partial\Pi^{\mp}(t)$ we have

$$\frac{d}{dt}\iint_{\Pi^{-}(t)}\boldsymbol{\Phi}\,da=\iint_{\Pi^{-}(t)}\dot{\boldsymbol{\Phi}}\,da+\int_{\partial\Pi^{-}(t)\cap C(t)}V\boldsymbol{\Phi}^{-}\,ds\,, \tag{9}$$

$$\frac{d}{dt}\iint_{\Pi^{+}(t)}\boldsymbol{\Phi}\,da=\iint_{\Pi^{+}(t)}\dot{\boldsymbol{\Phi}}\,da-\int_{\partial\Pi^{+}(t)\cap C(t)}V\boldsymbol{\Phi}^{+}\,ds\,. \tag{10}$$

From the relations (9) and (10) we obtain the following referential form of *the surface transport relation* valid for any piecewise smooth field $\boldsymbol{\Phi}(x,t)$ on Π in the presence of the singular curve $C(t)$:

$$\frac{d}{dt}\iint_{\Pi}\boldsymbol{\Phi}\,da=\iint_{\Pi}\dot{\boldsymbol{\Phi}}\,da-\int_{\Pi\cap C(t)}V[\boldsymbol{\Phi}]ds\,. \tag{11}$$

Let us now extend the surface divergence theorems (5) in the presence of the singular curve. As there is no time differentiation here, our discussion is confined to a fixed time.

For example, let the surface mixed 2nd-order tensor field $\boldsymbol{S}(x,t) \in E \otimes T_x M$ be piecewise smooth on any fixed $\Pi \subset M$ divided into two regular parts Π^- and Π^+ by the singular surface curve C. The outward unit normal $\boldsymbol{\nu}^-$ of the boundary $\partial\Pi^-$ satisfies $\boldsymbol{\nu}^- = \boldsymbol{\nu}_C$ at C and $\boldsymbol{\nu}^- = \boldsymbol{\nu}$ elsewhere on $\partial\Pi^-$. Likewise, the outward unit normal $\boldsymbol{\nu}^+$ of $\partial\Pi^+$ satisfies $\boldsymbol{\nu}^+ = -\boldsymbol{\nu}_C$ at C and $\boldsymbol{\nu}^+ = \boldsymbol{\nu}$ elsewhere on $\partial\Pi^+$. Applying the surface divergence theorem (5)$_2$ on the regular parts Π^- and Π^+ of Π, we obtain

$$\begin{aligned} \int_{\partial\Pi} \boldsymbol{S\nu}\, ds &= \int_{\partial\Pi^-} \boldsymbol{S\nu}^- ds + \int_{\partial\Pi^+} \boldsymbol{S\nu}^+ ds - \int_{\partial\Pi^- \cap C} \boldsymbol{S}^- \boldsymbol{\nu}^- ds - \int_{\partial\Pi^+ \cap C} \boldsymbol{S}^+ \boldsymbol{\nu}^+ ds \\ &= \int_{\partial\Pi^-} \boldsymbol{S\nu}\, ds + \int_{\partial\Pi^+} \boldsymbol{S\nu}\, ds - \int_{\partial\Pi^- \cap C} \boldsymbol{S}^- \boldsymbol{\nu}_C ds - \int_{\partial\Pi^+ \cap C} \boldsymbol{S}^+ (-\boldsymbol{\nu}_C) ds \\ &= \iint_{\Pi^-} Div\boldsymbol{S}\, da + \iint_{\Pi^+} Div\boldsymbol{S}\, da + \int_{\Pi \cap C} \left(\boldsymbol{S}^+ - \boldsymbol{S}^-\right) \boldsymbol{\nu}_C\, ds\,, \end{aligned} \tag{12}$$

so that

$$\int_{\partial\Pi} \boldsymbol{S\nu}\, ds = \iint_{\Pi} Div\boldsymbol{S}\, da + \int_{\Pi \cap C} [\boldsymbol{S}] \boldsymbol{\nu}_C\, ds\ . \tag{13}$$

Analogous arguments lead to the following extensions of *the surface divergence theorems* (5)$_1$ and (5)$_3$ in the presence of the singular surface curve:

$$\int_{\partial\Pi} \boldsymbol{t} \cdot \boldsymbol{\nu}\, ds = \iint_{\Pi} Div\boldsymbol{t}\, da + \int_{\Pi \cap C} [\boldsymbol{t}] \cdot \boldsymbol{\nu}_C\, ds\,, \tag{14}$$

$$\begin{aligned} \int_{\partial\Pi} \boldsymbol{a} \times \boldsymbol{S\nu}\, ds &= \iint_{\Pi} \left\{ \boldsymbol{a} \times \left(Div\boldsymbol{S}\right) + \operatorname{ax}\left(\boldsymbol{S}\left(Grad\,\boldsymbol{a}\right)^T - \left(Grad\,\boldsymbol{a}\right)\boldsymbol{S}^T \right) \right\} da \\ &\quad + \int_{\Pi \cap C} [\boldsymbol{a} \times \boldsymbol{S}] \boldsymbol{\nu}_C\, ds\ , \end{aligned} \tag{15}$$

for any vector fields $\boldsymbol{t}(x,t) \in T_x M$ and $\boldsymbol{a}(x,t) \in E$.

4 Compatibility conditions

Let the finite one-sided limits $\boldsymbol{\Phi}^{\mp}(x_C, t)$ be differentiable along $C(t)$. Then adopting the Hadamard lemma (see Truesdell and Toupin 1960, sec. 174, 175) to our case of the surface M with the moving singular curve $C(t)$, we have

$$\frac{d\boldsymbol{\Phi}^{\mp}}{ds} = \left(Grad\,\boldsymbol{\Phi}\right)^{\mp} \frac{d\boldsymbol{x}_C}{ds} = \left(Grad\,\boldsymbol{\Phi}\right)^{\mp} \boldsymbol{\tau}_C\ . \tag{16}$$

Subtracting the relation (16) with the $-$ sign from the one with the $+$ sign yields

$$\frac{d}{ds}[\boldsymbol{\Phi}] = [Grad\,\boldsymbol{\Phi}]\boldsymbol{\tau}_C = [(Grad\,\boldsymbol{\Phi})\boldsymbol{\tau}_C]\,, \tag{17}$$

which means that derivative along $C(t)$ of the jump is the jump of derivative along $C(t)$. The relation (17) is *the geometric compatibility conditions* for the field $\boldsymbol{\Phi}$ on M with $C(t)$.

But the Hadamard lemma is valid for any $(n-1)-$dimensional hypersurface in the $n-$dimensional space. Treating time t as an additional dimension, for our case of the singular curve $C(t)$ moving on the surface M we may also write

$$\frac{d\boldsymbol{\Phi}^{\mp}}{dt}=\frac{\partial\boldsymbol{\Phi}^{\mp}}{\partial t}+\left(Grad\,\boldsymbol{\Phi}\right)^{\mp}\frac{d\boldsymbol{x}_C}{dt}\,, \tag{18}$$

from which it follows that

$$\frac{d}{dt}[\boldsymbol{\Phi}]=[\frac{\partial\boldsymbol{\Phi}}{\partial t}]+[Grad\,\boldsymbol{\Phi}]\left(V\boldsymbol{\nu}_C+T\boldsymbol{\tau}_C\right). \tag{19}$$

The relation (19) can be called *the kinematic compatibility condition* for the field $\boldsymbol{\Phi}$ on the surface M with the moving singular curve $C(t)$.

In the important special case when $\boldsymbol{\Phi}$ is continuous on M (i.e. $[\boldsymbol{\Phi}]=\boldsymbol{0}$, or when $\boldsymbol{\Phi}$ is a scalar field and $[\boldsymbol{\Phi}]=const$) but $Grad\,\boldsymbol{\Phi}$ may exhibit a jump, from (17) it follows that $[Grad\,\boldsymbol{\Phi}]\boldsymbol{\tau}_C=\boldsymbol{0}$. Since according to (2), $[Grad\,\boldsymbol{\Phi}]\boldsymbol{n}_C=\boldsymbol{0}$, the general expression for $[Grad\,\boldsymbol{\Phi}]$ can only be of the form

$$[Grad\,\boldsymbol{\Phi}]\boldsymbol{\nu}_C=\boldsymbol{A}\quad\text{or}\quad[Grad\,\boldsymbol{\Phi}]=\boldsymbol{A}\otimes\boldsymbol{\nu}_C\,, \tag{20}$$

where the amplitude $\boldsymbol{A}$ belongs to the same space as the field $\boldsymbol{\Phi}$. As a result of (19) and (20) for the continuous field $\boldsymbol{\Phi}$ we obtain

$$[\frac{\partial\boldsymbol{\Phi}}{\partial t}]+V\boldsymbol{A}=\boldsymbol{0}. \tag{21}$$

This is *the kinematic compatibility condition* for the continuous field $\boldsymbol{\Phi}$ on M with $C(t)$.

Kinematics of the refined, resultant, thermodynamic shell model proposed by Pietraszkiewicz (2011) is described by two fields defined over the reference base surface M: the position vector $\boldsymbol{y}(x,t)$ representing the translatory motion of the moving base surface $M(t)$ and the proper orthogonal tensor $\boldsymbol{Q}(x,t)$ representing the energetic mean rotary motion of the shell cross sections. The corresponding translational $\boldsymbol{v}$ and angular $\boldsymbol{\omega}$ velocity vectors are defined by

$$\boldsymbol{v}=\frac{\partial\boldsymbol{y}}{\partial t}\,,\quad\boldsymbol{\omega}=\mathrm{ax}\left(\frac{\partial\boldsymbol{Q}}{\partial t}\boldsymbol{Q}^T\right),\quad\boldsymbol{\omega}\times\boldsymbol{Q}=\frac{\partial\boldsymbol{Q}}{\partial t}. \tag{22}$$

In most cases the shell motions $\boldsymbol{y}(x,t)$ and $\boldsymbol{Q}(x,t)$ are continuous fields on M with $C(t)$. Applying the relation (21) to such continuous motions we obtain

$$[\boldsymbol{v}]+V[\left(Grad\,\boldsymbol{y}\right)\boldsymbol{\nu}_C]=\boldsymbol{0}\,,\quad[\boldsymbol{\omega}\times\boldsymbol{Q}]+V[\left(Grad\,\boldsymbol{Q}\right)\boldsymbol{\nu}_C]. \tag{23}$$

We can recall that $Grad\,\boldsymbol{y} = \boldsymbol{F} \in E \otimes T_x M$ is the surface deformation gradient. We can also note that in (23)$_2$, $\operatorname{ax}\left\{\left(Grad\,\boldsymbol{Q}\right)\boldsymbol{Q}^T\right\} = \boldsymbol{K} \in E \otimes T_x M$ is the shell bending tensor. This allow us to write (23) more concisely as

$$[\boldsymbol{v}] + V[\boldsymbol{F}\boldsymbol{\nu}_C] = \boldsymbol{0}, \quad [\boldsymbol{\omega}] + V[\boldsymbol{K}\boldsymbol{\nu}_C]. \tag{24}$$

The relations (24) are *the kinematical compatibility conditions for the continuous motion* of the shell with a coherent singular curve $C(t)$, which within description of a phase transition phenomenon were formulated by Eremeyev and Pietraszkiewicz (2004) using alternative arguments.

The phase transition is the transformation of a thermodynamic system from one state of matter to another during which some changes may occur in the internal lattice structure of the body. In thin shells undergoing phase transition it may happen that while the translational motion is coherent at $C(t)$, the rotational motion may exhibit jump at $C(t)$ which results that the second of (24) may not be satisfied. In such a case the interface $C(t)$ is usually called *incoherent in rotations*.

5 Referential jump conditions for non-material singular thermomechanical processes in shells

In the refined, resultant thermomechanics of shells developed by Pietraszkiewicz (2011) three surface fields on M as independent field variables were used: the position vector $\boldsymbol{y}(x,t) \in E$ of $M(t)$ (or equivalently the translation vector $\boldsymbol{u}(x,t) = \boldsymbol{y}(x,t) - \boldsymbol{x}$ of M), the mean rotation tensor $\boldsymbol{Q}(x,t) \in \mathrm{Orth}^+$, and the mean referential absolute temperature $\theta(x,t) > 0$. The triad of fields $\{\boldsymbol{u}, \boldsymbol{Q}, \theta\}$ defines a thermomechanical process of the shell. In this refined thermomechanic shell model, the resultant 2D balance of energy was completed by an interstitial working flux vector field $\boldsymbol{w}(x,t) \in T_x M$. Then for any regular part $\Pi \subset M$ the corresponding resultant 2D laws of shell thermomechanics in the referential (Lagrangian) description - the balances of mass, linear and angular momenta, and energy as well as the entropy inequality - became exact resultant implications of corresponding 3D laws of rational thermomechanics proposed by Truesdell and Toupin (1960):

$$\frac{d}{dt}\int_\Pi \rho\, da - \iint_\Pi c\, da = 0\,, \tag{25}$$

$$\iint_\Pi \rho \boldsymbol{f}\, da - \frac{d}{dt}\iint_\Pi \boldsymbol{l}\, da + \int_{\partial\Pi \setminus \partial M_f} \boldsymbol{N}\boldsymbol{\nu}\, ds + \int_{\partial\Pi \cap \partial M_f} \boldsymbol{n}^*\, ds = \boldsymbol{0}\,, \tag{26}$$

$$\iint_{\Pi} \rho \boldsymbol{c}\, da - \frac{d}{dt}\iint_{\Pi} \boldsymbol{k}\, da + \iint_{\Pi} (\boldsymbol{y} + \rho \boldsymbol{f})\, da - \frac{d}{dt}\iint_{\Pi} (\boldsymbol{y} \times \boldsymbol{l})\, da$$
$$+ \int_{\partial\Pi \setminus \partial M_f} (\boldsymbol{M\nu} + \boldsymbol{y} \times \boldsymbol{N\nu})\, ds + \int_{\partial\Pi \cap \partial M_f} (\boldsymbol{m^*} + \boldsymbol{y} \times \boldsymbol{n^*})\, ds = \boldsymbol{0}\,, \tag{27}$$

$$\frac{d}{dt}\iint_{\Pi} \rho\varepsilon\, da - \iint_{\Pi} \left(\boldsymbol{N} \bullet \boldsymbol{E}^{\circ} + \boldsymbol{M} \bullet \boldsymbol{K}^{\circ}\right) da - \int_{\partial\Pi} \boldsymbol{w} \cdot \boldsymbol{\nu}\, ds$$
$$- \iint_{\Pi} \rho r\, da + \int_{\partial\Pi \setminus \partial M_h} \boldsymbol{q} \cdot \boldsymbol{\nu}\, ds - \int_{\partial\Pi \cap \partial M_h} q^*\, ds = 0\,, \tag{28}$$

$$\frac{d}{dt}\iint_{\Pi} \rho\eta\, da \ge \iint_{\Pi} \rho\left(\frac{r}{\theta} - s\right) da - \int_{\partial\Pi \setminus \partial M_h} \left(\frac{\boldsymbol{q}}{\theta} + \boldsymbol{s}\right) \cdot \boldsymbol{\nu}\, ds - \int_{\partial\Pi \cap \partial M_h} \left(\frac{q^*}{\theta^*} + s^*\right) ds\,. \tag{29}$$

In the resultant 2D balance laws of shell thermomechanics (25) - (28) the following 2D mechanical fields have been used: $\rho(x,t) > 0$ and $c(x,t)$ are the (referential) surface mass and mass production (densities), $\boldsymbol{f}(x,t)$ and $\boldsymbol{c}(x,t)$ are the force and couple vectors per unit mass of M, $\boldsymbol{l}(x,t) \in E$ and $\boldsymbol{k}(x,t) \in E$ are the surface linear momentum and angular momentum vectors per unit area of M, $\boldsymbol{N}(x,t) \in E \otimes T_x M$ and $\boldsymbol{M}(x,t) \in E \otimes T_x M$ are the referential surface stress resultant and stress couple tensors of Piola type with corresponding work-conjugate referential surface stretch $\boldsymbol{E}(x,t) \in E \otimes T_x M$ and bending $\boldsymbol{K}(x,t) \in E \otimes T_x M$ tensors, while $(.)^{\circ} = \boldsymbol{Q}\frac{d}{dt}\left(\boldsymbol{Q}^T(.)\right)$ is the co-rotational time derivative, respectively.

The resultant 2D energy balance (28) and the entropy inequality (29) are expressed through additional resultant surface fields: $\varepsilon(x,t)$ and $\eta(x,t)$ are the surface internal energy and entropy (densities), $r(x,t)$ and $s(x,t)$ are the surface heat and extra surface heat supply (densities) , all per unit mass of M, while $\boldsymbol{q}(x,t) \in T_x M$ and $\boldsymbol{s}(x,t) \in T_x M$ are the surface heat flux and extra entropy supply vectors per unit area of M, respectively.

In (26) - (29), ∂M_f and ∂M_h are those parts of ∂M along which the external resultant mechanical and heat fields are prescribed; the prescribed values of the fields are denoted by an asterisk $(.)^*$. Definitions of all resultant 2D surface fields entering (25) - (29) in terms of their 3D counterparts are given in Pietraszkiewicz (2011), where the local forms of 2D laws of shell thermomechanics are derived under assumption that M be a regular smooth surface and all the surface fields be smooth.

In the present report within any regular fixed $\Pi \subset M$ we allow a moving, non-material, singular curve $C(t)$ on which some fields appearing in (25) - (29) may not be differentiable, see section 3.

In the presence of $C(t)$, from (11) we have

$$\frac{d}{dt}\iint_{\Pi} \rho \, da = \iint_{\Pi} \dot{\rho} \, da - \int_{\Pi \cap C(t)} V[\rho] ds \ , \tag{30}$$

so that (25) yields the referential, local balance of mass and mass jump conditions

$$\dot{\rho} - c = 0 \ \text{ in } M \ , \quad V[\rho] = 0 \ \text{ across } C(t) \, . \tag{31}$$

When the singular curve $C(t)$ is admitted, in 2D balances of momenta (26) and (27) some terms containing time derivatives can be transformed with the help of the surface transport relation (11) as follows:

$$\begin{aligned} &\frac{d}{dt}\iint_{\Pi} \boldsymbol{l} \, da = \iint_{\Pi} \dot{\boldsymbol{l}} \, da + \int_{\Pi \cap C(t)} V[\boldsymbol{l}] \, ds \, , \\ &\frac{d}{dt}\iint_{\Pi} \left(\boldsymbol{k} + \boldsymbol{y} \times \boldsymbol{l}\right) da = \iint_{\Pi} \left(\dot{\boldsymbol{k}} + \dot{\boldsymbol{y}} \times \boldsymbol{l} + \boldsymbol{y} \times \dot{\boldsymbol{l}}\right) da \\ &\qquad + \int_{\Pi \cap C(t)} V[\boldsymbol{k} + \boldsymbol{y} \times \boldsymbol{l}] ds \, . \end{aligned} \tag{32}$$

To some other terms we apply the extended surface divergence theorems (13) and (15), which yields

$$\begin{aligned} &\int_{\partial \Pi} \boldsymbol{N} \boldsymbol{\nu} \, ds = \iint_{\Pi} Div \, \boldsymbol{N} \, da + \int_{\Pi \cap C(t)} [\boldsymbol{N}] \boldsymbol{\nu}_C ds \ , \\ &\int_{\partial \Pi} \boldsymbol{M} \boldsymbol{\nu} \, ds = \iint_{\Pi} Div \, \boldsymbol{M} \, da + \int_{\Pi \cap C(t)} [\boldsymbol{M}] \boldsymbol{\nu}_C ds \ , \\ &\int_{\partial \Pi} \boldsymbol{y} \times \boldsymbol{N} \boldsymbol{\nu} \, ds = \iint_{\Pi} \left\{ \boldsymbol{y} \times \left(Div \, \boldsymbol{N} \right) + \mathrm{ax}\left(\boldsymbol{N}\boldsymbol{F}^T + \boldsymbol{F}\boldsymbol{N}^T \right) \right\} da + \int_{\Pi \cap C(t)} [\boldsymbol{y} \times \boldsymbol{N}] \boldsymbol{\nu}_C ds \ . \end{aligned} \tag{33}$$

Introducing (32) and (33) into (26) and (27) we obtain the local referential balances of momenta and dynamic boundary conditions given in Pietraszkiewicz (2011), f. (30) and $(33)_{1,2}$, and additionally the following referential dynamic jump conditions across $C(t)$:

$$[\boldsymbol{N}]\boldsymbol{\nu}_C + V[\boldsymbol{l}] = \boldsymbol{0} \, , \quad [\boldsymbol{M}]\boldsymbol{\nu}_C + [\boldsymbol{y} \times \boldsymbol{N}]\boldsymbol{\nu}_C + V[\boldsymbol{l} + \boldsymbol{y} \times \boldsymbol{k}] = \boldsymbol{0} \, . \tag{34}$$

But in Section 1 we have already assumed that $M(t)$ remains the regular smooth base surface during the whole thermomechanical process. This means that $[\boldsymbol{y}] = \boldsymbol{0}$, and the second of dynamic jump conditions (34) can be reduced to

$$[\boldsymbol{M}]\boldsymbol{\nu}_C + V[\boldsymbol{k}] = \boldsymbol{0} \, . \tag{35}$$

Similarly, in the presence of the singular surface curve $C(t)$ some terms in the 2D energy balance (28) are transformed as follows:

$$\begin{aligned} &\frac{d}{dt}\iint_{\Pi} \rho\varepsilon \, da = \iint_{\Pi} \rho\dot{\varepsilon} \, da + \int_{\Pi \cap C(t)} V[\rho\varepsilon] \, ds \ , \\ &\int_{\partial \Pi} \boldsymbol{w} \cdot \boldsymbol{\nu} \, ds = \iint_{\Pi} Div \, \boldsymbol{w} \, da - \int_{\Pi \cap C(t)} [\boldsymbol{w}] \cdot \boldsymbol{\nu}_C ds \ , \\ &\int_{\partial \Pi} \boldsymbol{q} \cdot \boldsymbol{\nu} \, ds = \iint_{\Pi} Div \, \boldsymbol{q} \, da - \int_{\Pi \cap C(t)} [\boldsymbol{q}] \cdot \boldsymbol{\nu}_C ds \ . \end{aligned} \tag{36}$$

Introducing (36) into (28) we obtain again the referential, resultant, local balance of energy and the thermal boundary conditions (48) and $(34)_1$ of Pietraszkiewicz (2011), and additionally the referential energetic jump conditions across $C(t)$:

$$V[\rho\varepsilon] + [\boldsymbol{w} - \boldsymbol{q}] \cdot \boldsymbol{\nu}_C = 0 \ . \tag{37}$$

Finally, in the presence of the singular surface curve $C(t)$ some terms in the resultant 2D entropy inequality are transformed applying (11) and (14) as follows:

$$\begin{aligned} &\frac{d}{dt}\iint_{\Pi} \rho\eta\, da = \iint_{\Pi} \rho\dot{\eta}\, da + \int_{\Pi \cap C(t)} V[\rho\eta]\, ds \ , \\ &\int_{\partial\Pi} \left(\frac{\boldsymbol{q}}{\theta} + \boldsymbol{s}\right) \cdot \boldsymbol{\nu}\, ds = \iint_{\Pi} Div\left(\frac{\boldsymbol{q}}{\theta} + \boldsymbol{s}\right) da - \int_{\Pi \cap C(t)} [\frac{\boldsymbol{q}}{\theta} + \boldsymbol{s}] \cdot \boldsymbol{\nu}_C\, ds \ . \end{aligned} \tag{38}$$

Then (29) with (38) leads to the local referential forms of the resultant 2D entropy inequality and entropic boundary inequality given in Pietraszkiewicz (2011), eqs. (57) and (58), and additionally to the following referential entropic jump inequality across $C(t)$:

$$V[\rho\eta] - [\frac{\boldsymbol{q}}{\theta} + \boldsymbol{s}] \cdot \boldsymbol{\nu}_C \geq 0 \ . \tag{39}$$

In many solids mass is not created during the thermomechanical process, so that in the resultant shell thermomechanics $c \equiv 0$, $\dot{\rho} = 0$ and, hence, $\rho = \rho(x)$. In such case the resultant balances of mass (30) or $(31)_1$ are identically satisfied. Then $(31)_2$ requires that $[\rho] = 0$. The latter relation means that the referential energetic jump conditions and the entropic jump inequality across $C(t)$ can be reduced to

$$\rho V[\varepsilon] + [\boldsymbol{w} - \boldsymbol{q}] \cdot \boldsymbol{\nu}_C = 0 \quad , \quad \rho V[\eta] - [\frac{\boldsymbol{q}}{\theta} + \boldsymbol{s}] \cdot \boldsymbol{\nu}_C \geq 0. \tag{40}$$

Exactly these simplified 1D jump conditions have recently been derived by Konopińska and Pietraszkiewicz (2013).

Jump conditions formulated in this report are compatible only with the resultant shell thermomechanics proposed by Pietraszkiewicz (2011). But the surface transport relation (11), the surface divergence theorems (13) - (15) as well as several compatibility relations derived in section 4 can be utilized for formulation of corresponding jump conditions in other versions of 2D shell thermomechanics. Various such alternative versions were proposed for example by Naghdi (1972), Murdoch (1976), Zhilin (1976), Simmonds (1984, 2010), Makowski and Pietraszkiewicz (2004), Eremeyev and Zubov (2008), and Eremeyev and Pietraszkiewicz (2009, 2010).

An additional strain energy density associated with 1D curvilinear interface was used by Pietraszkiewicz et al. (2007) to model the generalized capillarity type phenomena during the phase transition in shells. In particular, when the line tension along the interface curve was assumed constant the 1D analogue of the Laplace equation was obtained, see also Eremeyev and Zubov (2008). Similar idea could be utilized in the theory of curvilinear dislocations in shells, where the line tension would describe an energy of tube surrounding the dislocation curve. For other phenomena which might be modeled by additional fields associated with the 2D interface in continuum mechanics see, for example, Podstrigach and Povstenko (1985), Grinfeld (1991), Zubov (1997), Gurtin (2000), and Gurtin et al. (2010).

6 Irregular shell base surface

In section 1, we have assumed that the shell base surface in the reference (undeformed) placement M and in the actual (deformed) placement $M(t) = \chi(M,t)$ be the regular smooth surfaces which geometry varies smoothly with the position vectors $\boldsymbol{x}$ of M and $\boldsymbol{y}$ of $M(t)$, respectively. But real shell structures are often irregular ones such that already their reference base surface may contain folds, branches, intersections etc., which results that the position surface derivatives or the curvatures be discontinuous along some surface curves or at some surface points. Additional mechanical design elements such as stepwise thickness changes, parts made of different materials, stiffeners, technological junctions, assembling inaccuracies etc. may also influence the real geometry of the shell base surface in the undeformed placement. The turbine blade with inside cooling channels along its length , see for example Sadowski and Godlewski (2012) or Han et al. (2013), is an example of a thin structure which may be modeled by a shell with thermal singular curve. All such 1D discontinuities are associated with material surface curves and points which remain to be material during an arbitrary thermodynamic process.

In this section we extend definition of the shell base surface and assume it to be the piecewise smooth surface M in the reference placement. This means that from now M is the union of a finite number of regular, smooth and connected (but not necessarily simply connected) surface elements M_k , $k = 1,2,...,K$, with edges ∂M_k joined together along spatial Jordan curves $\Gamma_a = \partial M_{k_1} \cap \partial M_{k_2} \cap ... \cap \partial M_{k_A}$, $k_1 \neq k_2 \neq ... \neq k_A$, $a = 1,2,...,A$, not necessarily closed, where two or more distinct M_k meet, see Fig. 2. The union Γ of all Γ_a together with their end points (vertices) c_p , $p = 1,2,...,P$, form a complex spatial network, see Makowski et al. (1999), where more detailed discussion of the irregular shell base surface is given.

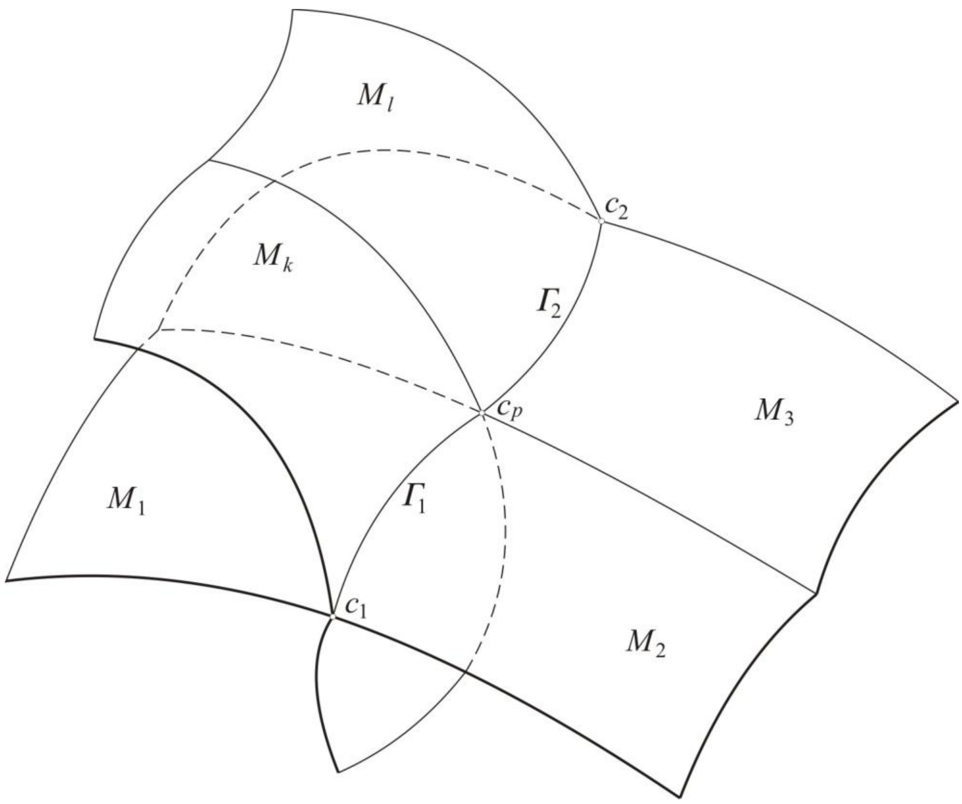

Figure 2. Example of the piecewise smooth surface M

The position vector $\boldsymbol{x}: M \to E$ of the piecewise smooth surface M is differentiable everywhere on M except on subsets of $\Gamma \subset M$ which area measure is zero. Each smooth part M_k of M can be oriented by a choice of unit normal vector $\boldsymbol{n}_k(x)$. The space of all vectors perpendicular to $\boldsymbol{n}_k(x)$ is then the tangent space $T_x M_k$ at each regular $x \in M_k$.

Consider first for simplicity a piecewise smooth surface M consisting of only two regular parts M_1 and M_2 joined together along the curve $\Gamma = \partial M_1 \cap \partial M_2$ parameterized by the length coordinate s, see Fig. 3. The curve Γ can be oriented consistently with the surface element M_1, if necessary. Then the unit tangent vector $\boldsymbol{\tau}_\Gamma = \partial \boldsymbol{x}_\Gamma / \partial s$ of Γ coincides with the unit tangent vector $\boldsymbol{\tau}_1$ of ∂M_1, and two other unit vectors $\boldsymbol{n}_\Gamma$ and $\boldsymbol{\nu}_\Gamma = \boldsymbol{\tau}_\Gamma \times \boldsymbol{n}_\Gamma$ along Γ are taken coinciding with $\boldsymbol{n}_1$ and $\boldsymbol{\nu}_1 = \boldsymbol{\tau}_1 \times \boldsymbol{n}_1$ of ∂M_1, respectively. Fig. 3 differs from Fig. 1 in that now the surface elements M_1 and M_2 at both sides of Γ have different orientations $\boldsymbol{n}_1 \neq \boldsymbol{n}_2$, and their outward unit normals $\boldsymbol{\nu}_1 \in T_x M_1$ and $\boldsymbol{\nu}_2 \in T_x M_2$ are elements of different 2D tangent spaces. In fact, along the curve Γ there is a fold of the irregular surface M.

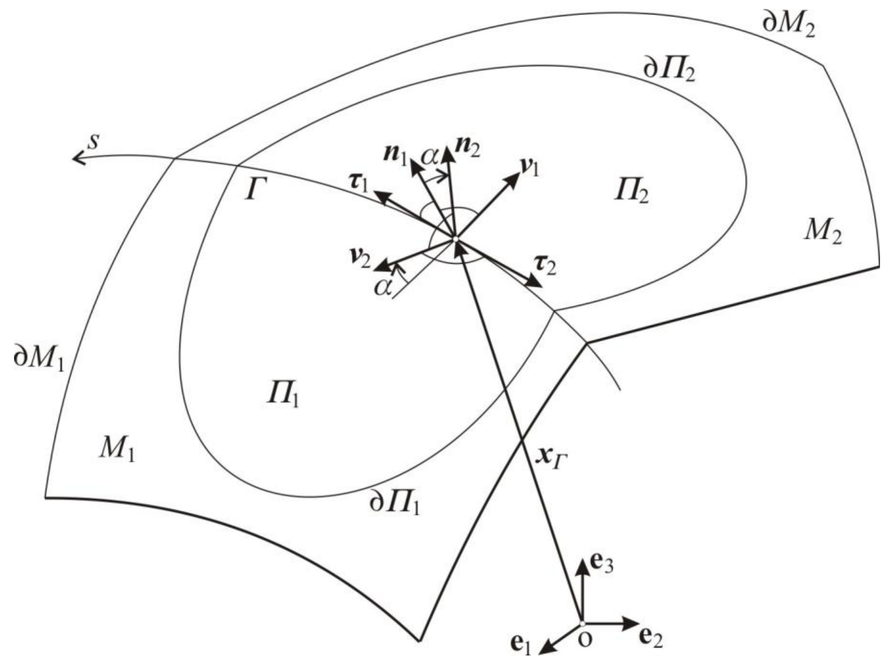

Figure 3. The irregular surface $M = M_1 \cup M_2$ with a fold Γ

Let the surface mixed 2nd-order tensor field $\boldsymbol{S}(\boldsymbol{x}) \in E \otimes T_x M$ be piecewise smooth on any $\Pi \subset M$ having a part of Γ in its interior, see Fig. 3. If Γ is oriented consistently with M_1, $\boldsymbol{\nu}_1$ of $\partial\Pi_1$ satisfies $\boldsymbol{\nu}_1 = \boldsymbol{\nu}_\Gamma$ along Γ and $\boldsymbol{\nu}_1 = \boldsymbol{\nu}$ elsewhere on $\partial\Pi_1$, but $\boldsymbol{\nu}_2$ of $\partial\Pi_2$ is not related to $\boldsymbol{\nu}_\Gamma$ along Γ, although still $\boldsymbol{\nu}_2 = \boldsymbol{\nu}$ elsewhere on $\partial\Pi_2$. Applying the surface divergence theorem $(5)_2$ separately on the regular parts Π_1 and Π_2 of Π, we obtain

$$\begin{aligned}\int_{\partial\Pi} \boldsymbol{S\nu}\, ds &= \int_{\partial\Pi_1} \boldsymbol{S\nu}_1\, ds + \int_{\partial\Pi_2} \boldsymbol{S\nu}_2\, ds - \int_{\partial\Pi_1\cap\Gamma} \boldsymbol{S}_1\boldsymbol{\nu}_1\, ds - \int_{\partial\Pi_2\cap\Gamma} \boldsymbol{S}_2\boldsymbol{\nu}_2\, ds \\ &= \int_{\partial\Pi_1} \boldsymbol{S\nu}_1\, ds + \int_{\partial\Pi_2} \boldsymbol{S\nu}_2\, ds - \int_{\Pi\cap\Gamma} \left(\boldsymbol{S}_1\boldsymbol{\nu}_1 + \boldsymbol{S}_2\boldsymbol{\nu}_2\right) ds \\ &= \iint_{\Pi} Div\,\boldsymbol{S}\, da - \int_{\Pi\cap\Gamma} [\![\boldsymbol{S\nu}]\!]\, ds\,,\end{aligned} \tag{41}$$

where now the jump at Γ is defined by

$$[\![\boldsymbol{S\nu}]\!] = \boldsymbol{S}_1\boldsymbol{\nu}_1 + \boldsymbol{S}_2\boldsymbol{\nu}_2\,. \tag{42}$$

Since M_1 and M_2 are obliquely joined along Γ, we can model geometry of such a junction by assigning an angle $\alpha(x_\Gamma)$ in the plane orthogonal to Γ in x_Γ, see Fig. 3. Then $\boldsymbol{\nu}_2 = -\boldsymbol{\nu}_\Gamma \cos\alpha$ and the jump $[\![.]\!]$ in (42) can explicitly be expressed as

$$[\![\boldsymbol{S\nu}]\!] = \boldsymbol{S}_1\boldsymbol{\nu}_\Gamma - \boldsymbol{S}_2\boldsymbol{\nu}_\Gamma \cos\alpha = -\left(\boldsymbol{S}_2\cos\alpha - \boldsymbol{S}_1\right)\boldsymbol{\nu}_\Gamma\,. \tag{43}$$

If there is no fold of M along Γ then $\alpha \equiv 0$ and $[\![\boldsymbol{S\nu}]\!] = -\left(\boldsymbol{S}_2 - \boldsymbol{S}_1\right)\boldsymbol{\nu}_\Gamma = -[\boldsymbol{S}]\boldsymbol{\nu}_\Gamma$, which brings (41) into full agreement with the extended surface divergence theorem (13).

It is apparent that using similar arguments we can extend also other patchwork surface divergence theorems on the piecewise smooth subsurface $\Pi = \Pi_1 \cup \Pi_2$, for example

$$\int_{\partial \Pi} \boldsymbol{t} \cdot \boldsymbol{\nu} ds = \iint_{\Pi} Div \boldsymbol{t} da - \int_{\Pi \cap \Gamma} [\![\boldsymbol{t} \cdot \boldsymbol{\nu}]\!] ds , \tag{44}$$

$$\begin{aligned} \int_{\partial \Pi} \boldsymbol{a} \times \boldsymbol{S} \boldsymbol{\nu} ds = & \iint_{\Pi} \left\{ \boldsymbol{a} \times \left(Div \boldsymbol{S} \right) + \mathrm{ax} \left(\boldsymbol{S} \left(Grad \boldsymbol{a} \right)^T - \left(Grad \boldsymbol{a} \right) \boldsymbol{S}^T \right) \right\} da \\ & - \int_{\Pi \cap \Gamma} [\![\boldsymbol{a} \times \boldsymbol{S} \boldsymbol{\nu}]\!] ds \ , \end{aligned} \tag{45}$$

for any surface vector fields $\boldsymbol{t}(x,t) \in T_x M$ and $\boldsymbol{a}(x,t) \in E$. Somewhat similar arguments were applied by Kazakevičiūtė-Makovska (2010) in her derivation of the patchwork surface divergence theorems used to formulate equilibrium problems of hybrid structures composed of interacting membranes and strings.

In more general cases each of the network curves Γ may represent a junction of more than two regular surfaces M_k as in branching and self-intersecting shells, see Konopińska and Pietraszkiewicz (2007). Let for example $\Gamma = \partial M_1 \cap \partial M_2 \cap \ldots \cap \partial M_K$ represent a junction of K regular, smooth and connected surface elements, and let $\Pi \subset M$ have a part of such Γ in its interior. Then the surface divergence theorem (5) applied separately on the regular parts $\Pi_1, \Pi_2, \ldots, \Pi_K$ of Π to the piecewise smooth mixed tensor field $\boldsymbol{S}(\boldsymbol{x}) \in E \otimes T_x M$ would lead to similar transformations as indicated in (41), only now the jump should be defined by

$$[\![\boldsymbol{S} \boldsymbol{\nu}]\!] = \sum_{k=1}^{K} \boldsymbol{S}_k \boldsymbol{\nu}_k \ . \tag{46}$$

Thus, the extended patchwork surface divergence theorems on the piecewise smooth surface comprising a part of Γ will have the same forms as (41), (44) and (45), only now with the jump $[\![.]\!]$ defined by the extended formula (46).

However, in case of branching and self-intersecting shells, the direct through-the-thickness integration requires special attention in order to assure the exact equivalence of 1D equilibrium conditions of the junction region. According to procedure proposed by Konopińska and Pietraszkiewicz (2007), each of the regular parts of 3D shell is first extended into the junction region up to the junction curve of the shell base surface. By such extension some fictitious tractions become automatically applied on surface strips, and through-the-thickness integration becomes performed twice within tubes in the junction region. To assure exact balances of forces and couples acting on the irregular shell base surface, some statically equivalent system of forces and couples has to be subtracted along the junction curves and at points of their intersection. As a result, this procedure suggests that in case of irregular shell structures it may be necessary to apply some completed resultant 2D laws of shell thermomechanics and more complex jump conditions than those given in section 5 with jumps defined by (46). For such completed resultant thermomechanic shell relations, each set corresponding to a particular type of shell irregularity should be discussed separately.

7 Conclusions

We have presented some geometric and kinematic relations associated with the surface curve moving on the shell base surface. For the piecewise smooth tensor fields acting on the shell base surface, the extended surface transport relation and the extended surface divergence theorems have been derived taking account of the moving singular surface curve. This has allowed to complete the referential, refined, resultant balance laws and entropy inequality of shell thermomechanics proposed in Pietraszkiewicz (2011) by the corresponding jump conditions at moving non-material singular surface curves.

Stationary (material) singular surface curves can model various irregularities of shell structures such as branching, intersections, abrupt changes of geometric and/or material parameters etc. It has been noted that in case of shell intersections the jumps appearing in the extended surface divergence theorems become more complex. In such cases the through-the-thickness integration procedure requires to take account of additional non-balanced forces and couples along the surface curve of irregularity.

Acknowledgements The research reported in this paper was supported by the National Centre of Science of Poland with the grant DEC – 2012/05/D/ST8/02298.

References

Abeyaratne, R. & Knowles, J.K. (2006). *Evolution of Phase Transitions: A Continuum Theory*. Cambridge: Cambridge Univ. Press.

Berezovski, A., Engelbrecht, J. & Maugin, G.A. (2008). *Numerical Simulations of Waves and Fronts in Inhomogeneous Solids*. New Jersey: World Scientific

Bernadou, M. & Cubier, A. (1998). Numerical analysis of junctions between shells. *Computer Methods in Applied Mechanics and Engineering* 161, 349-387.

Bhattacharya, K. (2003). *Microstructure of Martensite: Why It Forms and How It Gives Rise to the Shape-Memory Effect*. Oxford: Oxford Univ. Press.

Chróścielewski, J. (1996). *Family of C^0Finite Elements in the Non-linear Six-parameter Theory of Shells* (in Polish). Zesz. Nauk PG Nr 540, Bud. Ląd. LIII, 1-291. Gdańsk, Wydawnictwo Politechniki Gdańskiej.

Chróścielewski, J., Makowski, J. & Stumpf, H. (1997). Finite-element analysis of smooth, folded and multi-shell structures. *Computer Methods in Applied Mechanics and Engineering* 41, 1-47.

Chróścielewski, J., Makowski, J. & Pietraszkiewicz, W. (2002). Non-linear dynamics of flexible shell structures. *Computer Assisted Mechanics and Engineering Sciences* 9, 341-357.

Chróścielewski, J., Makowski, J. & Pietraszkiewicz, W. (2004). *Statics and Dynamics of Multi-Shells: Non-Linear Theory and the Finite Element Method*. Warsaw: Institute of Fundamental Technological Research Press (in Polish).

Chróścielewski, J., Konopińska, V. & Pietraszkiewicz, W. (2011). On modelling and non-linear elasto-plastic analysis of thin shells with deformable junctions. *Journal of Applied Mathematics and Mechanics – ZAMM* 91(6), 477-484.

Dunn, J.E. & Serrin, J. (1985). On the thermodynamics of interstitial working. *Archives for Rational Mechanics and Analysis* 85, 95-133.

Eremeyev, V.A. & Pietraszkiewicz, W. (2004). The nonlinear theory of elastic shells with phase transitions. *Journal of Elasticity* 74(1), 67-86.

Eremeyev, V.A. & Pietraszkiewicz, W. (2009). Phase transitions in thermoelastic and thermoviscoelastic shells. *Archives of Mechanics* 61, 125-152.

Eremeyev, V.A. & Pietraszkiewicz, W. (2010). Thermomechanics of shells undergoing phase transitions. *Journal of the Mechanics and Physics of Solids* 59, 1395-1412.

Eremeyev, V.A. & Zubov, L.M. (2008). *Mechanics of Elastic Shells*. Moscow: Nauka (in Russian).

Grinfeld, M. (1991). *Thermodynamics Methods in the Theory of Heterogeneous Systems*. Harlow: Longmans.

Gurtin, M.E. (2000). *Configurational Forces as Basic Concepts in Continuum Mechanics.* New York et al.: Springer.

Gurtin, M.E., Fried, E. & Anand, L. (2010). *Mechanics and Thermodynamics of Continua*. Cambridge: Cambridge Univ. Press.

Gurtin, M.E. & Murdoch, A.I. (1975). A continuum theory of elastic material surfaces. *Archives for Rational Mechanics and Analysis* 57, 291-323.

Kazakevičiūtė-Makowska, R. (2010). Statics of hybrid structures composed of interacting membranes and strings. *International Journal of Non-Linear Mechanics* 45, 186-192.

Konopińska, V. & Pietraszkiewicz, W. (2007). Exact resultant equilibrium conditions in the non-linear theory of branching and self-intersecting shells. *International Journal of Solids and Structures* 44, 352-369.

Konopińska, V. & Pietraszkiewicz, W. (2013). On jump conditions at non-material singular curves in the resultant shell thermomechanics. In W. Pietraszkiewicz & J. Górski (Eds.), *Shell Structures: Theory and Applications*, Vol. 3, London: CRC / Balkema (in print).

Kosiński, W. (1986). *Field Singularities and Wave Analysis in Continuum Mechanics.* Warsaw: Polish Sci. Publishers and Chicester: Ellis Horwood.

Makowski, J. & Pietraszkiewicz, W. (2002). *Thermomechanics of Shells with Singular Curves*. Zesz. Nauk. IMP PAN 528 (1487). Gdańsk: Wydawnictwo IMP PAN.

Makowski, J., Pietraszkiewicz, W. & Stumpf, H. (1998). On the general form of jump conditions for thin irregular shells. *Archives of Mechanics* 50, 483-495.

Makowski, J., Pietraszkiewicz, W. & Stumpf, H. (1999). Jump conditions in the non-linear theory of thin irregular shells. *Journal of Elasticity* 54, 1-26.

Makowski, J. & Stumpf, H. (1994). *Mechanics of Irregular Shell Structures*. Mitt. Inst. f. Mech. Nr 95. Bochum: Ruhr-Universität.

Murdoch, A.I. (1976). A thermodynamical theory of elastic material surfaces. *Quarterly Journal of Mechanics and Applied Mathematics* 29, 245-275.

Pietraszkiewicz, W. (2001). On using rotations as primary variables in the non-linear theory of thin irregular shells. In D. Durban et al. (Eds.), *Advances in the Mechanics of Plates and Shells*, pp. 245-254. New York et al.: Kluver Acad. Publ.

Pietraszkiewicz, W. (2011). Refined resultant thermomechanics of shells. *International Journal of Engineering Science* 49, 1112-1124.

Pietraszkiewicz, W. & Konopińska, V. (2011). On unique kinematics for the branching shells. *International Journal of Solids and Structures* 48, 2238-2244.

Pietraszkiewicz, W., Eremeyev, V.A. & Konopińska, V. (2007). Extended non-linear relations of elastic shells undergoing phase transitions. *Zeitschrift für Angewandte Mathematik und Mechanik* 87(2), 150-159.

Podstrigach, Ya.S., & Povstenko, Yu.Z. (1985). *Introduction to Mechanics of Surface Phenomena in Deformable Bodies* (in Russian). Kiev: Naukova Dumka.

Simmonds, J.G. (1984). The nonlinear thermodynamical theory of shells: Descent from 3-dimensions without thickness expansion. In: E.L. Axelrad & F.A. Emmerling (Eds.), *Flexible Shells*, pp. 1-11. Berlin, Springer-Verlag.

Simmonds, J.G. (2011). A classical, nonlinear thermodynamic theory of elastic shells based on a single constitutive assumption. *Journal of Elasticity* 105(1-2), SI, 305-312.

Teng, J.G. (2004). Shell junctions. In: J.G. Teng & J.M. Rotter (Eds.), *Buckling of Thin Metal Structures*, Chapter 3, pp. 369-408. London: Taylor & Francis.

Šilhavý. M. (1997). *The Mechanics and Thermodynamics of Continuous Media.* Berlin: Springer.

Truesdell, C. & Noll, W. (1965). The Non-Linear Field Theories of Mechanics. In S. Flügge (Ed.), *Handbuch der Physik*, Band III/3. Berlin-Heidelberg-NewYork: Springer-Verlag.

Truesdell, C. & Toupin, R. (1960). The Classical Field Theories. In S. Flügge (Ed.), *Handbuch der Physik*, Band III/1. Berlin: Springer-Verlag.

Zhilin, P.A. (1976). Mechanics of deformable directed surfaces. *International Journal of Solids and Structures* 12(9-10), 635-648.

Zubov, L.M. (1997). *Nonlinear Theory of Dislocations and Disclinations in Elastic Bodies.* Berlin et al.: Springer-Verlag.