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The resultant linear six-field theory of elastic shells: What it brings to the classical linear shell models?

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Basic relations of the resultant linear six-field theory of shells are established by consistent linearization of the resultant 2D non-linear theory of shells. As compared with the classical linear shell models of Kirchhoff-Love and Timoshenko-Reissner type, the six-field linear shell model contains the drilling rotation as an independent kinematic variable as well as two surface drilling couples with two work-conjugate surface drilling bending measures are present in description of the shell stress-strain state. Among new results obtained here within the six-field linear theory of elastic shells there are: 1) formulation of the extended static-geometric analogy; 2) derivation of complex BVP for complex independent variables; 3) description of deformation of the shell boundary element; 4) the Cesáro type formulas and expressions for the vectors of stress functions along the shell boundary contour; 5) discussion on explicit appearance of gradients of 2D stress and strain measures in the resultant stress working.

1 Introduction

The classical linear theory of thin elastic shells, usually called of the Kirchhoff-Love type, was initiated by Love [1], developed in thousands of papers and summarized in many monographs, for example by Love [2], Gol'denveiser [3], Naghdi [4], Green and Zerna [5], Başar and Krätzig [6], or Novozhilov et al. [7], to mention only a few. The refined linear theory of elastic shells with an additional account of transverse shear deformations, here called of the Timoshenko-Reissner type, was extensively treated for example by Naghdi [8], Librescu [9], Pelekh [10] or Reddy [11]. Almost all linear versions of shell theory are based on various approximations in kinematical description of linear elasticity following from a through-the-thickness polynomial expansion with truncation at some level, asymptotic analyses, applying kinematic constraints etc., see Fig. 1, the left graph. In all classical linear shell models the resultant couple stress vectors do not have the drilling (normal to the shell midsurface) components due to identification of deformed and undeformed position vectors of linear elasticity in definition of the resultant couple stress vectors. The drilling component (about the normal) of the linearized rotation vector does not appear as the kinematical field variable, and there is no drilling components of the 2D bending vectors as well.

In the shell model proposed by Reissner [12] dynamics of stress resultants and stress couples together with the concept of virtual work were the basic notions. In this approach the equilibrium equations of the deformed shell were first derived by *exact* through-the-thickness integration of the equilibrium equations of continuum mechanics, see Fig. 1, the middle graph. In this approach, the resultant 2D stress and couple vectors

$\boldsymbol{n}^{\alpha}, \boldsymbol{m}^{\alpha}$ satisfying the resultant equilibrium equations of the deformed shell each had three independent components in any vector base. Then the 2D virtual work identity allowed one to construct *uniquely* the 2D shell kinematics consisting of the translation vector $\boldsymbol{u}$ and the rotation tensor $\boldsymbol{Q}$ (or an equivalent finite rotation vector $\boldsymbol{\psi}$), which were the through-the-thickness energetic averages of the non-linear displacement distribution of the shell cross section. The corresponding 2D stretch and bending vectors $\boldsymbol{\varepsilon}_{\alpha}, \boldsymbol{\kappa}_{\alpha}$ defined by $\boldsymbol{u}, \boldsymbol{\psi}$ each had three independent non-vanishing components in any vector base. This approach was developed in a number of papers and summarized in the books by Libai and Simmonds [13], Chróścielewski et al. [14] and Eremeyev and Zubov [15]. Such non-linear resultant six-field shell model could be consistently linearized for small translations, rotations and 2D strain measures. As a result, this led to *different* formulation of the linear shell theory than the ones presented in all classical papers and books mentioned above. The main difference consisted in that now also the drilling rotation (about the normal) remained as an independent kinematic variable, and two drilling couples with two work-conjugate drilling bending measures appeared in the description of stress and strain states of the linear shell theory.

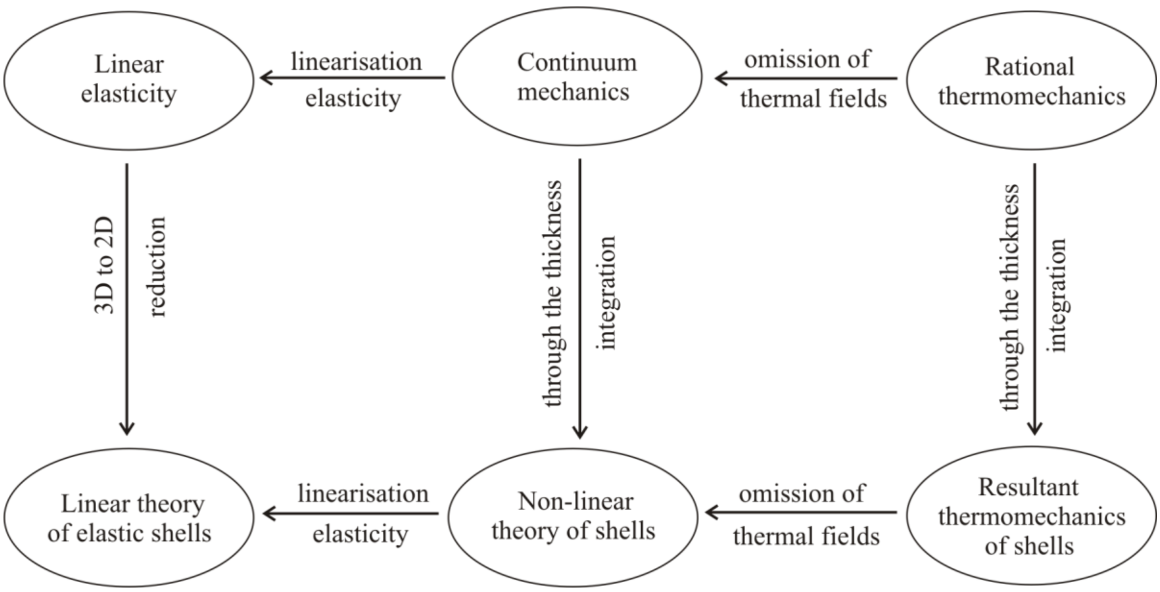

Figure 1. Formulations of the linear theory of elastic shells

Along the same line of reasoning, we may ask what happens when we begin with the 3D non-linear continuum thermomechanics, reduce it consistently to a 2D non-linear shell thermomechanics, then omit temperature effects, assume the elastic material behaviour and linearize all the shell relations, see Fig. 1, the right graph. Pietraszkiewicz [16] derived the resultant 2D balance laws of mass, linear and angular momenta, and energy as well as the entropy inequality by direct through-the-thickness integration of corresponding 3D laws of rational thermomechanics of Truesdell and Toupin [17]. It was found that the resultant mechanical power could not be expressed entirely through the resultant 2D stress and strain measures, because the through-the-thickness integration process did not allow to account for the mechanical power following from the stresses acting on surfaces parallel to the shell base surface as well as from self-equilibrated parts of distribution of stresses and body forces through the shell cross section. Thus, an additional 2D mechanical power called an interstitial working had to be added to the resultant balance of energy. Only then the so refined resultant 2D balance of energy and entropy inequality could be regarded as the *exact resultant implication* of 3D continuum thermomechanics. However, even for a thermoelastic material the 2D restrictions imposed by the procedure of Coleman and Noll [18] upon the constitutive equations of such resultant refined 2D shell thermomechanics allowed some 2D fields to depend also on surface gradients of the shell strain measures and on higher surface gradients of temperature. The possible dependence of constitutive equations upon surface gradients of the strain measures does not disappear even after linearization of all shell relations and omission of temperature effects. This feature contradicts all forms of constitutive equations appearing in the classical linear versions of elastic shells.

In this paper we wish to investigate in more detail what the resultant six-field linear theory of isotropic elastic shells brings to the well established classical linear shell models. In section 2 we recall some exact 2D relations of the resultant theory of shells which are needed in what follows. The exact relations are then consistently linearized in section 3 under small translations and rotations of the shell base surface and for the isotropic elastic material. This leads to the complete BVP of the resultant six-fields linear theory of isotropic elastic shells. As compared with the classical linear shell models of K-L and T-R type, the six-field shell model additionally contains the drilling rotation as an independent kinematic variable as well as two surface drilling couples and two work-conjugate surface drilling bending measures. For the six-field shell model the extended static-geometric analogy is established in section 4 and the corresponding complex formulation of BVP for complex independent variables is proposed.

Deformation of the shell boundary element is described in section 5. In particular, the total rotation vector of the boundary element is found by the superposition of two rotations: the one corresponding to the global linearized rotation and the additional rotation following from the stretch along the shell boundary contour. The Cesáro type formulas for the translation and rotation vectors are derived and expressions for the vectors of stress functions are established along the shell boundary contour. The final section 6 contains discussion on contribution of surface gradients of 2D shell strain and stress measures to the resultant 2D stress working.

2 Notation and some exact resultant shell relations

Let us recall some exact resultant relations of the non-linear theory of shells, see for example Libai and Simmonds [13], Chróścielewski et al. [14], Eremeyev and Pietraszkiewicz [19], or Pietraszkiewicz and Konopińska [20].

A shell is a three-dimensional (3D) solid body identified in a reference (undeformed) placement with a region B of the physical space $\mathcal{E}$ having the translation vector space E. The shell boundary $\partial\mathrm{B}$ consists of three separable parts: the upper M^{+} and lower M^{-} shell faces, and the lateral shell boundary surface $\partial\mathrm{B}^{*}$. The position vectors $\mathbf{x}$ and $\mathbf{y}=\chi(\mathbf{x})$ of any material particle in the reference and deformed placements, respectively, can conveniently be represented by

$$\mathbf{x}=\boldsymbol{x}+\xi\boldsymbol{n}, \quad \mathbf{y}=\boldsymbol{y}(\boldsymbol{x})+\boldsymbol{\zeta}(\boldsymbol{x},\xi), \quad \boldsymbol{\zeta}(\boldsymbol{x},0)=\mathbf{0}. \tag{1}$$

Here $\boldsymbol{x}$ and $\boldsymbol{y}$ are the position vectors of some shell base surface M and $N=\chi(M)$ in the reference and deformed placements, respectively, ξ is the distance from M along the unit normal vector $\boldsymbol{n}$ orienting M such that $\xi\in[-h^{-},h^{+}]$, $h=h^{-}+h^{+}$ is the shell thickness, $\boldsymbol{\zeta}$ is a deviation vector of $\mathbf{y}$ from N, while χ and χ mean the 3D and 2D deformation functions, respectively. In what follows we use the convention that fields defined on the shell base surface are written by italic symbols, except in a few explicitly defined cases.

Within the resultant non-linear theory of shells formulated in the referential description, the respective 2D internal contact stress resultant $\boldsymbol{n}_\nu$ and stress couple $\boldsymbol{m}_\nu$ vectors, defined at the edge ∂R of an arbitrary part of the deformed base surface $R=\chi(P), P\subset M$, but measured per unit length of the undeformed edge ∂P having the outward unit normal vector $\boldsymbol{\nu}$, are defined by

$$\begin{aligned}\boldsymbol{n}_\nu &= \int_-^+ \mathbf{P}\mathbf{n}^* \mu \mathrm{d}\xi = \boldsymbol{n}^\alpha \nu_\alpha, \quad \boldsymbol{n}^\alpha = \int_-^+ \mathbf{p}^\alpha \mu \mathrm{d}\xi, \quad \int_-^+ \equiv \int_{-h^-}^{+h^+}, \\ \boldsymbol{m}_\nu &= \int_-^+ \boldsymbol{\zeta}\times\mathbf{P}\mathbf{n}^* \mu \mathrm{d}\xi = \boldsymbol{m}^\alpha \nu_\alpha, \quad \boldsymbol{m}^\alpha = \int_-^+ \boldsymbol{\zeta}\times\mathbf{p}^\alpha \mu \mathrm{d}\xi.\end{aligned} \tag{2}$$

Here $\mathbf{P}=\mathbf{p}^\varphi\otimes\mathbf{g}_\varphi+\mathbf{p}^3\otimes\mathbf{g}_3$ is the Piola stress tensor in the shell space, $\mathbf{g}_i=\partial\mathbf{x}/\partial\theta^i$, $i=1,2,3,$ are the base vectors in B, $\mathbf{n}^*=\mathbf{g}^\alpha\nu_\alpha$, $\alpha=1,2,$ is the external normal to the reference shell orthogonal cross section $\partial\mathrm{P}^*$, $\mathbf{p}^\alpha=\delta^\alpha_\varphi\mathbf{p}^\varphi$, $\nu_\alpha=\boldsymbol{\nu}\cdot\boldsymbol{a}_\alpha$, $\boldsymbol{a}_\alpha=\partial\boldsymbol{x}/\partial\theta^\alpha$ are the base vectors of M, and μ^α_φ are geometric shifters with $\mu=\det\left(\mu^\alpha_\varphi\right)$, see Naghdi [4] or Pietraszkiewicz [21].

The resultant 2D equilibrium equations satisfied for any part $P\subset M$ are

$$\boldsymbol{n}^\alpha|_\alpha+\boldsymbol{f}=\mathbf{0}, \quad \boldsymbol{m}^\alpha|_\alpha+\boldsymbol{y},_\alpha\times\boldsymbol{n}^\alpha+\boldsymbol{m}=\mathbf{0}, \tag{3}$$

where $(\cdot)|_\alpha$ is the covariant derivative in the metric of M, while $\boldsymbol{f}$ and $\boldsymbol{m}$ are the external resultant surface force and couple vectors applied at N, but measured per unit area of M.

The resultant dynamic boundary conditions satisfied along ∂M_f are

$$\boldsymbol{n}^\alpha\nu_\alpha=\boldsymbol{n}^*, \quad \boldsymbol{m}^\alpha\nu_\alpha=\boldsymbol{m}^*, \tag{4}$$

where $\boldsymbol{n}^*,\boldsymbol{m}^*$ are the external resultant boundary force and couple vectors applied along $\partial N_f=\chi(\partial M_f)$ but measured per unit length of ∂M_f, and $(\cdot)^*$ means the prescribed field.

In order the conditions (3) and (4) be satisfied, the resultant fields $\boldsymbol{n}^\alpha$ and $\boldsymbol{m}^\alpha$ require a unique 2D shell kinematics associated with the shell base surface M. Applying the virtual work identity Libai and Simmonds [22, 13], Chróścielewski et al. [23, 14], and Eremeyev and Pietraszkiewicz [19] proved that such 2D kinematics consists of the translation vector $\boldsymbol{u}$ and the proper orthogonal (rotation) tensor $\boldsymbol{Q}$, both describing the gross deformation (work-averaged through the shell thickness) of the shell cross section, such that

$$\boldsymbol{y}=\boldsymbol{x}+\boldsymbol{u}, \quad \boldsymbol{t}_\alpha=\boldsymbol{Q}\boldsymbol{a}_\alpha, \quad \boldsymbol{t}=\boldsymbol{Q}\boldsymbol{n}, \tag{5}$$

where $\boldsymbol{t}_\alpha,\boldsymbol{t}$ are three directors attached to any point of $N=\chi(M)$. Thus, along the complementary boundary contour $\partial M_d=\partial M\setminus\partial M_f$ the kinematic boundary conditions are

$$\boldsymbol{u}=\boldsymbol{u}^*, \quad \boldsymbol{Q}=\boldsymbol{Q}^*. \tag{6}$$

The vectors $\boldsymbol{n}^\alpha,\boldsymbol{m}^\alpha$ and $\boldsymbol{f},\boldsymbol{c}$ can naturally be expressed in components relative to the rotated base $\boldsymbol{t}_\beta,\boldsymbol{t}$ by

$$\begin{aligned}\boldsymbol{n}^\alpha &= N^{\alpha\beta}\boldsymbol{t}_\beta+Q^\alpha\boldsymbol{t}, \quad \boldsymbol{m}^\alpha=\boldsymbol{t}\times M^{\alpha\beta}\boldsymbol{t}_\beta+M^\alpha\boldsymbol{t}=\varepsilon_{\lambda\beta}M^{\alpha\lambda}\boldsymbol{t}^\beta+M^\alpha\boldsymbol{t}, \\ \boldsymbol{f} &= f^\beta\boldsymbol{t}_\beta+f\boldsymbol{t}, \quad \boldsymbol{m}=\boldsymbol{t}\times m^\beta\boldsymbol{t}_\beta+m\boldsymbol{t}=\varepsilon_{\lambda\beta}m^\lambda\boldsymbol{t}^\beta+m\boldsymbol{t},\end{aligned} \tag{7}$$

where $\varepsilon_{\alpha\beta}$ are components of the skew surface permutation tensor. The 2D components M^{α} are usually called the drilling couples.

The shell strain $\boldsymbol{\varepsilon}_{\alpha}$ and bending $\boldsymbol{\kappa}_{\alpha}$ vectors associated with the 2D shell kinematics (5), which are work-conjugate to the respective stress resultant $\boldsymbol{n}^{\alpha}$ and stress couple $\boldsymbol{m}^{\alpha}$ vectors, are defined by

$$\begin{aligned} \boldsymbol{\varepsilon}_{\alpha} &= \boldsymbol{y},_{\alpha} - \boldsymbol{t}_{\alpha} = \boldsymbol{u},_{\alpha} + (\mathbf{1} - \boldsymbol{Q})\boldsymbol{a}_{\alpha} = E_{\alpha\beta}\boldsymbol{t}^{\beta} + E_{\alpha}\boldsymbol{t}\,, \\ \boldsymbol{\kappa}_{\alpha} &= \operatorname{ax}\left(\boldsymbol{Q},_{\alpha}\boldsymbol{Q}^{T}\right) = \boldsymbol{t} \times K_{\alpha\beta}\boldsymbol{t}^{\beta} + K_{\alpha}\boldsymbol{t} = \varepsilon_{\lambda\beta}K_{\alpha}^{\cdot\lambda}\boldsymbol{t}^{\beta} + K_{\alpha}\boldsymbol{t}\,, \end{aligned} \tag{8}$$

where $\mathbf{1}$ is the metric tensor of 3D space and $\operatorname{ax}(\cdot)$ is the axial vector of skew tensor $(\cdot)$. The 2D components K_{α} can be called the drilling bending measures.

Since $\boldsymbol{u}$ (or $\boldsymbol{y}$) and $\boldsymbol{Q}$ are independent kinematic variables, it is required that for the simply connected regular surface M,

$$\varepsilon^{\alpha\beta}\boldsymbol{y}_{|\alpha\beta} = \mathbf{0}\,, \quad \varepsilon^{\alpha\beta}\boldsymbol{Q}_{|\alpha\beta} = \mathbf{0}\,. \tag{9}$$

From $(9)_1$ and $(8)_1$ it follows that

$$\begin{aligned} \boldsymbol{y}_{|\alpha\beta} &= \boldsymbol{\varepsilon}_{\alpha|\beta} + \boldsymbol{t}_{\alpha|\beta} = \boldsymbol{\varepsilon}_{\alpha|\beta} + (\boldsymbol{Q}\boldsymbol{a}_{\alpha})_{|\beta} = \boldsymbol{\varepsilon}_{\alpha|\beta} + \boldsymbol{Q},_{\beta}\boldsymbol{Q}^{T}\boldsymbol{Q}\boldsymbol{a}_{\alpha} + \boldsymbol{Q}\boldsymbol{a}_{\alpha|\beta} \\ &= \boldsymbol{\varepsilon}_{\alpha|\beta} + \boldsymbol{\kappa}_{\beta} \times \boldsymbol{t}_{\alpha} + b_{\alpha\beta}\boldsymbol{t}\,, \end{aligned} \tag{10}$$

and the condition $(9)_1$ is equivalent to

$$\varepsilon^{\alpha\beta}\left(\boldsymbol{\varepsilon}_{\alpha|\beta} + \boldsymbol{\kappa}_{\beta} \times \boldsymbol{t}_{\alpha}\right) = \mathbf{0}\,. \tag{11}$$

To reveal the meaning of $(9)_2$, we note that

$$\boldsymbol{Q},_{\alpha} = \boldsymbol{\kappa}_{\alpha} \times \boldsymbol{Q}\,, \quad \boldsymbol{Q}_{|\alpha\beta} = \boldsymbol{\kappa}_{\alpha|\beta} \times \boldsymbol{Q} + \boldsymbol{\kappa}_{\alpha} \times \boldsymbol{Q},_{\beta} = \left[\boldsymbol{\kappa}_{\alpha|\beta} \times \mathbf{1} + \boldsymbol{\kappa}_{\alpha} \times (\boldsymbol{\kappa}_{\beta} \times \mathbf{1})\right]\boldsymbol{Q}\,. \tag{12}$$

Then using the identity $\varepsilon^{\alpha\beta}\boldsymbol{\kappa}_{\alpha} \times (\boldsymbol{\kappa}_{\beta} \times \boldsymbol{v}) = 1/2\,\varepsilon^{\alpha\beta}(\boldsymbol{\kappa}_{\alpha} \times \boldsymbol{\kappa}_{\beta}) \times \boldsymbol{v}$ valid for any vector field $\boldsymbol{v} \in E$, and noting that $\boldsymbol{Q}$ is non-singular, the condition $(9)_2$ with (12) becomes equivalent to

$$\varepsilon^{\alpha\beta}\left(\boldsymbol{\kappa}_{\alpha|\beta} + \frac{1}{2}\boldsymbol{\kappa}_{\alpha} \times \boldsymbol{\kappa}_{\beta}\right) = \mathbf{0}\,. \tag{13}$$

The relations (11) and (13) are the compatibility conditions for the vectorial shell strain measures $\boldsymbol{\varepsilon}_{\alpha}, \boldsymbol{\kappa}_{\alpha}$ in the resultant six-field non-linear theory of shells. Similar conditions for the non-material shell base surface were first proposed by Libai and Simmonds [22, 13]. In lines of principal curvatures of the shell base surface M these conditions become equivalent to those proposed by Reissner [12].

It is sometime more convenient to describe rotations by a finite rotation vector $\boldsymbol{\psi}$ in place of $\boldsymbol{Q}$. By the Euler theorem the rotation tensor $\boldsymbol{Q}$ can be expressed in terms of an angle of rotation ϕ about an axis described by the eigenvector $\boldsymbol{i}$ corresponding to the real eigenvalue $+1$ of $\boldsymbol{Q}$ such that

$$\boldsymbol{Q}\boldsymbol{i} = +\boldsymbol{i}\,, \quad \cos\phi = \frac{1}{2}\left(\operatorname{tr}\boldsymbol{Q} - 1\right), \quad \sin\phi = \frac{1}{2}\left(\boldsymbol{Q} - \boldsymbol{Q}^{T}\right). \tag{14}$$

Introducing the finite rotation vector $\boldsymbol{\psi} = \sin\phi\,\boldsymbol{i}$, we have

$$\boldsymbol{Q} = \mathbf{1} + \boldsymbol{\psi} \times \mathbf{1} + \frac{1}{2\cos^{2}\phi/2}\left(\boldsymbol{\psi} \times \mathbf{1}\right)^{2}\,. \tag{15}$$

For an arbitrary deformation of the shell space, distribution of displacements through the shell thickness is non-linear, in general. To account for this non-linearity, Chróścielewski et al. [14] and Pietraszkiewicz et al. [24] introduced the intrinsic deviation vector $\mathbf{e}(\theta^\alpha,\xi)$ defined by

$$\mathbf{e}=\boldsymbol{Q}^T\boldsymbol{\zeta}-\xi\boldsymbol{n}=e^\rho(\xi)\mathbf{g}_\rho(\xi)+e(\xi)\boldsymbol{n}\ ,\tag{16}$$

where $\boldsymbol{Q}\mathbf{e}$ is a measure of deviation of the deformed curved material fiber from its linear rotated shape $\xi\boldsymbol{Q}\boldsymbol{n}$, see Fig. 2.

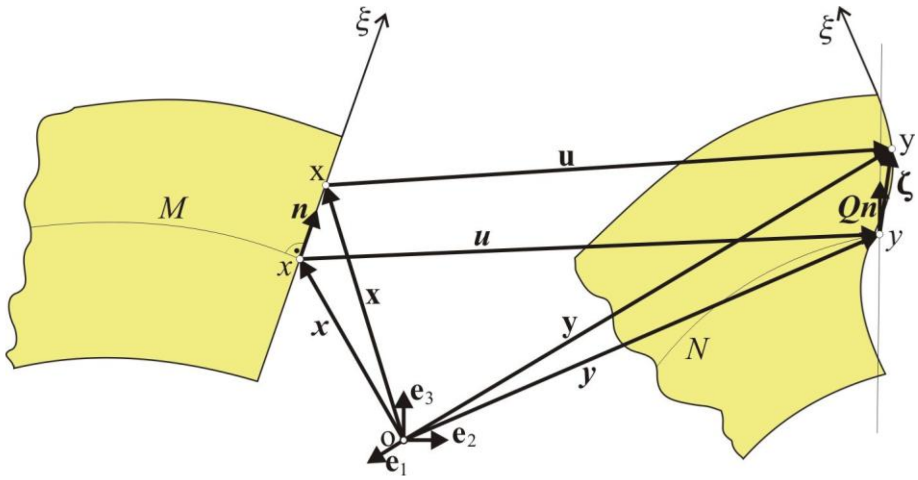

Figure 2. Finite deformation of the shell cross section

In the non-linear theory of shells the 3D deformation gradient of the shell space $\mathbf{F}=\mathbf{y},_i\otimes\mathbf{g}^i$ can be decomposed according to the modified polar decomposition as $\mathbf{F}=\boldsymbol{Q}\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}\neq\boldsymbol{\Lambda}^T$ is the modified 3D stretch tensor. Since $\mathbf{y}=\boldsymbol{y}+\boldsymbol{\zeta}$, we have

$$\begin{aligned}&\mathbf{y},_\alpha=\boldsymbol{y},_\alpha+\left[\boldsymbol{Q}(\xi\boldsymbol{n}+\mathbf{e})\right],_\alpha=\boldsymbol{t}_\alpha+\boldsymbol{\varepsilon}_\alpha+\boldsymbol{\kappa}_\alpha\times\left[\boldsymbol{Q}(\xi\boldsymbol{n}+\mathbf{e})\right]+\boldsymbol{Q}(\xi\boldsymbol{n},_\alpha+\mathbf{e},_\alpha),\\&\mathbf{y},_3=\boldsymbol{Q}(\boldsymbol{n}+\mathbf{e},_3),\end{aligned}\tag{17}$$

$$\mathbf{F}=\boldsymbol{Q}\left\{\left[\boldsymbol{x},_\alpha+\boldsymbol{\varepsilon}_\alpha+\boldsymbol{\kappa}_\alpha\times(\xi\boldsymbol{n}+\mathbf{e})+(\xi\boldsymbol{n},_\alpha+\mathbf{e},_\alpha)\right]\otimes\mathbf{g}^\alpha+(\boldsymbol{n}+\mathbf{e},_3)\otimes\boldsymbol{n}\right\}.\tag{18}$$

From (18) it follows that

$$\begin{aligned}&\boldsymbol{\Lambda}=\left[\boldsymbol{x},_\alpha+\boldsymbol{\varepsilon}_\alpha+\boldsymbol{\kappa}_\alpha\times(\xi\boldsymbol{n}+\mathbf{e})+(\xi\boldsymbol{n},_\alpha+\mathbf{e},_\alpha)\right]\otimes\mathbf{g}^\alpha+(\boldsymbol{n}+\mathbf{e},_3)\otimes\boldsymbol{n},\\&\dot{\boldsymbol{\Lambda}}=\left[\dot{\boldsymbol{\varepsilon}}_\alpha+\dot{\boldsymbol{\kappa}}_\alpha\times(\xi\boldsymbol{n}+\mathbf{e})+\boldsymbol{\kappa}_\alpha\times\dot{\mathbf{e}}+\dot{\mathbf{e}},_\alpha\right]\otimes\mathbf{g}^\alpha+\dot{\mathbf{e}},_3\otimes\boldsymbol{n},\end{aligned}\tag{19}$$

where overdot means the material time derivative.

With the results (18) and (19) the 3D stress power density per unit undeformed volume $\Sigma=(\mathbf{FS}):\dot{\mathbf{F}}$, where $\mathbf{S}=\mathbf{s}^\alpha\otimes\mathbf{g}_\alpha+\mathbf{s}^3\otimes\boldsymbol{n}$ is the symmetric 2nd Piola-Kirchhoff stress tensor, can be transformed as follows:

$$\Sigma=(\boldsymbol{Q}\boldsymbol{\Lambda}\mathbf{S}):(\dot{\boldsymbol{Q}}\boldsymbol{\Lambda}+\boldsymbol{Q}\dot{\boldsymbol{\Lambda}})=(\boldsymbol{\Lambda}\mathbf{S}\boldsymbol{\Lambda}^T):(\boldsymbol{Q}^T\dot{\boldsymbol{Q}})+(\boldsymbol{\Lambda}\mathbf{S}):(\boldsymbol{Q}^T\boldsymbol{Q}\dot{\boldsymbol{\Lambda}})=(\boldsymbol{\Lambda}\mathbf{S}):\dot{\boldsymbol{\Lambda}},\tag{20}$$

because the double-dot (scalar) product $:$ of the symmetric $\boldsymbol{\Lambda}\mathbf{S}\boldsymbol{\Lambda}^T$ and skew $\boldsymbol{Q}^T\dot{\boldsymbol{Q}}$ tensors vanishes in the space of 2nd-order tensors $E\otimes E$.

The resultant 2D stress power density $\mathit{\Sigma}$ per unit area of M can be defined by direct through-the-thickness integration of Σ,

$$\mathit{\Sigma}=\int_-^+(\boldsymbol{\Lambda}\mathbf{S}):\dot{\boldsymbol{\Lambda}}\,\mu\,\mathrm{d}\xi\,.\tag{21}$$

Let us introduce (19) into (21) and perform appropriate transformations, which leads to (see [24], Eqs (28) and (29))

$$\Sigma = \mathbf{n}^{\alpha} \cdot \dot{\boldsymbol{\varepsilon}}_{\alpha} + \mathbf{m}^{\alpha} \cdot \dot{\boldsymbol{\kappa}}_{\alpha} + \int_{-}^{+} (\dot{\mathbf{e}} \times \boldsymbol{\Lambda}\mathbf{s}^{\alpha}) \mu \mathrm{d}\xi \cdot \boldsymbol{\kappa}_{\alpha} + \int_{-}^{+} (\boldsymbol{\Lambda}\mathbf{s}^{\alpha}) \cdot \dot{\mathbf{e}},_{\alpha} \mu \mathrm{d}\xi + \int_{-}^{+} (\boldsymbol{\Lambda}\mathbf{s}^{3}) \cdot \dot{\mathbf{e}},_{3} \mu \mathrm{d}\xi \,, \tag{22}$$

where

$$\begin{aligned} \mathbf{n}^{\alpha} &= \boldsymbol{Q}^{T} \boldsymbol{n}^{\alpha} = \int_{-}^{+} \boldsymbol{\Lambda}\mathbf{s}^{\alpha} \, \mu \mathrm{d}\xi = N^{\alpha\beta} \boldsymbol{a}_{\beta} + Q^{\alpha} \boldsymbol{n} \,, \\ \mathbf{m}^{\alpha} &= \boldsymbol{Q}^{T} \boldsymbol{m}^{\alpha} = \int_{-}^{+} (\xi \boldsymbol{n} + \mathbf{e}) \times \boldsymbol{\Lambda}\mathbf{s}^{\alpha} \, \mu \mathrm{d}\xi = \varepsilon_{\lambda\beta} M^{\alpha\lambda} \boldsymbol{a}^{\beta} + M^{\alpha} \boldsymbol{n} \,, \end{aligned} \tag{23}$$

$$\boldsymbol{\varepsilon}_{\alpha} = \boldsymbol{Q}^{T} \boldsymbol{\varepsilon}_{\alpha} = E_{\alpha\beta} \boldsymbol{a}^{\beta} + E_{\alpha} \boldsymbol{n} \,, \quad \boldsymbol{\kappa}_{\alpha} = \boldsymbol{Q}^{T} \boldsymbol{\kappa}_{\alpha} = \varepsilon_{\lambda\beta} K_{\alpha}^{\cdot\lambda} \boldsymbol{a}^{\beta} + K_{\alpha} \boldsymbol{n} \,. \tag{24}$$

The 2D relation (22) with (23) and (24) is the *exact resultant implication* of the 3D stress power density Σ in the non-linear six-field theory of shells. The first two terms of (22) represent the 2D effective stress power density expressible entirely in terms of the 2D resultant stress measures and corresponding work-conjugate 2D strain measures. The last three integrals in (22) represent the additional part of the 2D stress power density which is not expressible through the resultant 2D shell stress and strain measures alone.

3 The linear six-field shell theory

In the linear theory of shells it is assumed that translations and rotations are small:

$$\varepsilon = \max_{x \in M} \left(\| \boldsymbol{u} \| , h \| \boldsymbol{\psi} \| \right) << 1 \,. \tag{25}$$

In order to derive the linear theory of shells from the non-linear continuum mechanics one needs two steps: contraction of dimension from 3 to 2 and linearization. These two steps can be performed in different order. In Chróścielewski et al. [14] it was explicitly shown that if one first *linearizes* the equilibrium equations of continuum mechanics and then *contracts* the dimension, one always obtains the 5-field linear shell theory of T-R type. When the equilibrium equations of continuum mechanics are linearized, geometry of the deformed continuum is usually identified with that in the undeformed placement. As a result, in the linearized integral definition $(2)_2$ of the surface stress couple vectors $\boldsymbol{m}^{\alpha}$ instead of $\boldsymbol{\zeta}$ the rectilinear vector $\xi \boldsymbol{n}$ normal to the *undeformed* base surface is vectorialy multiplied by the linearized stress vector $\mu \sigma^{\alpha i} \mathbf{g}_i$ of linear elasticity leading to their component representations $\boldsymbol{m}^{\alpha} = \varepsilon_{\lambda\beta} M^{\alpha\lambda} \boldsymbol{a}^{\beta}$. Thus, the drilling component of $\boldsymbol{m}^{\alpha}$ along $\boldsymbol{n}$ is cut off by definition. Then, from the virtual work identity the drilling rotation component of $\boldsymbol{\psi}$ on $\boldsymbol{n}$ remains indefinite, which finally leads to the 5-field linear shell theory of T-R type.

However, when the dimension contraction (through-the-thickness integration) of the non-linear continuum mechanics is first performed as in definition $(2)_2$ of $\boldsymbol{m}^{\alpha}$, the deviation vector $\boldsymbol{\zeta}$ still follows the curved deformed material fibre, which is neither rectilinear nor normal to the deformed shell base surface, see Fig. 2. The through-the-thickness integration $(2)_2$ leads then to three independent vector components of $\boldsymbol{m}^{\alpha}$, so that we have always two drilling couples M^{α} as components of $\boldsymbol{m}^{\alpha}$ relative to $\boldsymbol{t}$. Then the virtual work arguments require $\boldsymbol{Q}$ (or $\boldsymbol{\psi}$) to have *three* independent rotational scalar parameters, and after linearization the component of $\boldsymbol{\psi}$ on $\boldsymbol{n}$ remains non-zero, in general. As a result, this leads to the complete six-field linear shell theory containing also the drilling rotation ψ, two drilling couples M^{α} and two work-conjugate drilling bending measures K_{α}. This is the reason why we have to derive the relations of the linear shell theory by the consistent linearization of corresponding 2D relations of the resultant six-field non-linear shell theory.

Linearizing all kinematic relations with regard to small $\boldsymbol{u}$ and $\boldsymbol{\psi}$, we obtain

$$\sin\phi \approx \phi\,, \quad \cos\phi \approx 1\,, \quad \boldsymbol{\psi} \approx \phi \boldsymbol{i}\,, \quad \boldsymbol{Q} \approx \boldsymbol{1} + \boldsymbol{\psi} \times \boldsymbol{1}\,, \tag{26}$$

$$\begin{aligned} &\boldsymbol{u} = u_\alpha \boldsymbol{a}^\alpha + w\boldsymbol{n}\,, \quad \boldsymbol{\psi} = \boldsymbol{n} \times (\psi_\alpha \boldsymbol{a}^\alpha) + \psi\boldsymbol{n} = \varepsilon^{\alpha\beta}\psi_\alpha \boldsymbol{a}_\beta + \psi\boldsymbol{n} = \frac{1}{2}\left(\boldsymbol{a}^\alpha \times \boldsymbol{t}_\alpha + \boldsymbol{n} \times \boldsymbol{t}\right), \\ &\boldsymbol{t}_\alpha = \boldsymbol{a}_\alpha + \boldsymbol{\psi} \times \boldsymbol{a}_\alpha = (a_{\alpha\beta} + \varepsilon_{\alpha\beta}\psi)\boldsymbol{a}^\beta + \psi_\alpha \boldsymbol{n}\,, \quad \boldsymbol{t} = \boldsymbol{n} + \boldsymbol{\psi} \times \boldsymbol{n} = \boldsymbol{n} + \psi_\alpha \boldsymbol{a}^\alpha\,, \end{aligned} \tag{27}$$

$$\begin{aligned} &\boldsymbol{u},_\alpha = (u_{\beta|\alpha} - b_{\alpha\beta}w)\boldsymbol{a}^\beta + (w,_\alpha + b_{\alpha\beta}u^\beta)\boldsymbol{n}\,, \\ &\boldsymbol{\psi},_\alpha = (\varepsilon_{\lambda\beta}\psi^\lambda{}_{|\alpha} - b_{\alpha\beta}\psi)\boldsymbol{a}^\beta + (\psi,_\alpha - \varepsilon_{\lambda\beta}b^\lambda_\alpha \psi^\beta)\boldsymbol{n}\,. \end{aligned} \tag{28}$$

In the six-field linear theory of shells the vectorial strain measures (8) reduce to

$$\boldsymbol{\varepsilon}_\alpha = \boldsymbol{u},_\alpha - \boldsymbol{\psi} \times \boldsymbol{a}_\alpha = E_{\alpha\beta}\boldsymbol{a}^\beta + E_\alpha \boldsymbol{n}\,, \quad \boldsymbol{\kappa}_\alpha = \boldsymbol{\psi},_\alpha = \varepsilon_{\lambda\beta}K_\alpha^{\cdot\lambda}\boldsymbol{a}^\beta + K_\alpha \boldsymbol{n}\,, \tag{29}$$

$$\begin{aligned} &E_{\alpha\beta} = \boldsymbol{u},_\alpha \cdot \boldsymbol{a}_\beta - \varepsilon_{\alpha\beta}\boldsymbol{\psi} \cdot \boldsymbol{n} = u_{\beta|\alpha} - b_{\alpha\beta}w - \varepsilon_{\alpha\beta}\psi\,, \\ &E_\alpha = \boldsymbol{u},_\alpha \cdot \boldsymbol{n} + \varepsilon_{\alpha\beta}\boldsymbol{\psi} \cdot \boldsymbol{a}^\beta = w,_\alpha + b^\beta_\alpha u_\beta + \psi_\alpha\,, \end{aligned} \tag{30}$$

$$\begin{aligned} &K_{\alpha\beta} = \boldsymbol{\psi},_\alpha \cdot \varepsilon_{\beta\lambda}\boldsymbol{a}^\lambda = \psi_{\beta|\alpha} - \varepsilon_{\beta\lambda}b^\lambda_\alpha \psi\,, \\ &K_\alpha = \boldsymbol{\psi},_\alpha \cdot \boldsymbol{n} = \psi,_\alpha - \varepsilon_{\lambda\beta}b^\lambda_\alpha \psi^\beta\,. \end{aligned} \tag{31}$$

Please note that indeed in the six-field linear shell theory the kinematic relations (29) - (31) contain also components of the drilling rotation ψ and the drilling bending measures K_α. In the classical linear shell theories the components ψ and K_α do not appear by definition in the kinematic relations.

With definitions (29) - (31) the vector compatibility conditions (11) and (13) reduce to

$$\varepsilon^{\alpha\beta}\left(\boldsymbol{\varepsilon}_{\alpha|\beta} + \boldsymbol{\kappa}_\beta \times \boldsymbol{a}_\alpha\right) = \boldsymbol{0}\,, \quad \varepsilon^{\alpha\beta}\boldsymbol{\kappa}_{\alpha|\beta} = \boldsymbol{0}\,, \tag{32}$$

which in components relative to $\boldsymbol{a}^\lambda, \boldsymbol{n}$ read

$$\begin{aligned} &\varepsilon^{\alpha\beta}\left(E_{\alpha\lambda|\beta} - E_\alpha b_{\beta\lambda} + \varepsilon_{\alpha\lambda}K_\beta\right) = 0\,, \quad \varepsilon^{\alpha\beta}\left(E_{\alpha|\beta} + E_{\alpha\lambda}b^\lambda_\beta + K_{\alpha\beta}\right) = 0\,, \\ &\varepsilon^{\alpha\beta}\left(\varepsilon^{\rho\lambda}K_{\alpha\rho|\beta} + b^\lambda_\alpha K_\beta\right) = 0\,, \quad \varepsilon^{\alpha\beta}\left(K_{\alpha|\beta} + \varepsilon^{\lambda\rho}K_{\alpha\lambda}b_{\beta\rho}\right) = 0\,. \end{aligned} \tag{33}$$

In the linear six-field shell theory the base vectors $\boldsymbol{t}_\beta, \boldsymbol{t}$ in $(7)_2$ can be approximated by $\boldsymbol{a}_\beta, \boldsymbol{n}$, and the component representations of $\boldsymbol{n}^\alpha$, $\boldsymbol{m}^\alpha$ and $\boldsymbol{f}$, $\boldsymbol{m}$ given in (7) reduce to

$$\begin{aligned} &\boldsymbol{n}^\alpha = N^{\alpha\beta}\boldsymbol{a}_\beta + Q^\alpha \boldsymbol{n}\,, \quad \boldsymbol{m}^\alpha = \varepsilon_{\lambda\beta}M^{\alpha\lambda}\boldsymbol{a}^\beta + M^\alpha \boldsymbol{n}\,, \\ &\boldsymbol{f} = f^\beta \boldsymbol{a}_\beta + f\boldsymbol{n}\,, \quad \boldsymbol{m} = \varepsilon_{\lambda\beta}m^\lambda \boldsymbol{a}^\beta + m\boldsymbol{n}\,. \end{aligned} \tag{34}$$

Hence, the equilibrium equations (3) written trough components (34) in the base $\boldsymbol{a}_\beta, \boldsymbol{n}$ become

$$\begin{aligned} &N^{\alpha\beta}|_\alpha - b^\beta_\alpha Q^\alpha + f^\beta = 0\,, \quad Q^\alpha|_\alpha + b_{\alpha\beta}N^{\alpha\beta} + f = 0\,, \\ &M^{\alpha\beta}|_\alpha - Q^\beta + \varepsilon^{\lambda\beta}b_{\alpha\lambda}M^\alpha + m^\beta = 0\,, \quad M^\alpha|_\alpha + \varepsilon_{\alpha\beta}(N^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta}) + m = 0. \end{aligned} \tag{35}$$

Within such six-field linear shell theory the six linear equilibrium equations $(35)_{3,4}$ contain also the drilling couples M^α. In the classical linear shell theories of K-L and T-R type the components M^α do not appear in analogous shell relations.

The kinematic and dynamic structures of the linear six-field shell model are similar to those of the linear Cosserat surface discussed by Günther [25], but physical interpretations of fields of the two approaches are entirely different.

Twelve components of the resultant 2D stress measures in (35) and twelve corresponding work-conjugate components of the 2D strain measures require some constitutive equations refined above those following from the consistent 1st approximation to the strain energy density of isotropic elastic shells. Pietraszkiewicz and Konopińska [20] constructed the consistent 2nd approximation to the complementary energy density of the geometrically non-linear theory of isotropic elastic shells. From the density followed the refined constitutive equations for the 2D strain measures,

$$
\begin{aligned}
E_{\alpha\beta} &= \frac{1}{h} A_{\alpha\beta\lambda\mu}\left(N^{\lambda\mu} - b^{\lambda}_{\rho} M^{\rho\mu} + b^{\kappa}_{\kappa} M^{\lambda\mu} \right) - \frac{1}{h} b^{\kappa}_{\alpha} A_{\kappa\beta\lambda\mu} M^{\lambda\mu} , \\
K_{\alpha\beta} &= \frac{1}{h} A_{\alpha\beta\lambda\mu}\left(\frac{12}{h^2} M^{\lambda\mu} - b^{\lambda}_{\rho} N^{\rho\mu} + b^{\kappa}_{\kappa} N^{\lambda\mu} \right) - \frac{1}{h} b^{\kappa}_{\alpha} A_{\kappa\beta\lambda\mu} N^{\lambda\mu} , \\
E_{\alpha} &= \frac{4}{\alpha_s h} A_{\alpha 3\lambda 3} Q^{\lambda} , \\
A_{\alpha\beta\lambda\mu} &= \frac{1}{2E}\left[(1+\nu)\left(a_{\alpha\lambda} a_{\beta\mu} + a_{\alpha\mu} a_{\beta\lambda} \right) - 2\nu a_{\alpha\beta} a_{\lambda\mu} \right], \quad A_{\alpha 3\lambda 3} = \frac{1+\nu}{2E} a_{\alpha\lambda} ,
\end{aligned}
\tag{36}
$$

where E is the Young modulus, ν is the Poisson ratio, $\alpha_s = 5/6$, and h is the shell thickness. When inverted in lines of principal midsurface curvatures, the constitutive equations (36) led to the following physical components of the resultant 2D stress measures:

$$
\begin{aligned}
N_{11} &= C\left(E_{11} + \nu E_{22} \right) - D\left(\frac{1}{R_1} - \frac{1}{R_2} \right) K_{11} , \\
N_{12} &= \frac{1}{2} C(1-\nu)\left(E_{12} + E_{21} \right) - D(1-\nu)\left(\frac{1}{R_1} - \frac{1}{R_2} \right) K_{12} , \\
N_{21} &= \frac{1}{2} C(1-\nu)\left(E_{12} + E_{21} \right) + D(1-\nu)\left(\frac{1}{R_1} - \frac{1}{R_2} \right) K_{21} , \\
N_{22} &= C\left(E_{22} + \nu E_{11} \right) + D\left(\frac{1}{R_1} - \frac{1}{R_2} \right) K_{22} , \\
Q_1 &= \frac{1}{2} \alpha_t C(1-\nu) E_1 , \quad Q_2 = \frac{1}{2} \alpha_t C(1-\nu) E_2 ,
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
M_{11} &= D\left(K_{11} + \nu K_{22} \right) - D\left(\frac{1}{R_1} - \frac{1}{R_2} \right) E_{11} , \\
M_{12} &= \frac{1}{2} D(1-\nu)\left(K_{12} + K_{21} \right) - D(1-\nu)\left(\frac{1}{R_1} - \frac{1}{R_2} \right) E_{12} , \\
M_{21} &= \frac{1}{2} D(1-\nu)\left(K_{12} + K_{21} \right) + D(1-\nu)\left(\frac{1}{R_1} - \frac{1}{R_2} \right) E_{21} , \\
M_{22} &= D\left(K_{22} + \nu K_{11} \right) + D\left(\frac{1}{R_1} - \frac{1}{R_2} \right) E_{22} , \\
M_1 &= \alpha_d D(1-\nu) K_1 , \quad M_2 = \alpha_d D(1-\nu) K_2 ,
\end{aligned}
\tag{38}
$$

where

$$C = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \quad \alpha_d = \frac{4}{15}, \tag{39}$$

and R_1 and R_2 are the principal radii of midsurface curvatures.

When solving a linear shell problem with a regular base surface M, the drilling couples M^{α} are usually much smaller than the stress couples $M^{\alpha\beta}$, see estimates given in [20]. Similarly, the drilling bending measures K_{α} are much smaller than $K_{\alpha\beta}$. This is the reason why omission of the drilling components in the classical linear models of elastic shells does not lead to loss of accuracy of solutions within the first approximation to the shell strain energy.

However, many real shell structures may contain folds, branches, intersections, stiffeners, junctions with beams and columns and other engineering design elements, which allows one to form complicated spatial shell structures. In such irregular structures the kinematic junction conditions require three translations and three linearized rotations to be adjusted along the junctions of regular shell parts. In development of FEM design codes finite elements based on the six-field shell model have six dof in any node and allow for only C^0 interelement continuity. Such finite elements are numerically much more efficient than the finite elements based on the classical K-L type linear shell model requiring C^1 interelement continuity, see discussion in [14]. This is the reason why development of the linear six-field theory of elastic shells and discussion of its predictive capabilities seem to be important for engineering applications.

4 Static – geometric analogy and complex shell relations

Changing some indices and performing elementary algebraic transformations, the compatibility conditions (33) can be presented in the following equivalent forms :

$$\begin{aligned}
&(-\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}E_{\rho\lambda})_{|\alpha} - (-\varepsilon^{\beta\lambda}K_{\lambda}) + \varepsilon^{\alpha\beta}b_{\alpha\lambda}(-\varepsilon^{\lambda\rho}E_{\rho}) = 0, \\
&(-\varepsilon^{\alpha\rho}E_{\rho})_{|\alpha} + \varepsilon_{\alpha\beta}(\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}K_{\rho\lambda}) - \varepsilon_{\alpha\beta}b^{\alpha}_{\lambda}(-\varepsilon^{\lambda\rho}\varepsilon^{\kappa\beta}E_{\rho\kappa}) = 0, \\
&(\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}K_{\rho\lambda})_{|\alpha} - b^{\beta}_{\alpha}(-\varepsilon^{\alpha\rho}K_{\rho}) = 0, \\
&(-\varepsilon^{\alpha\rho}K_{\rho})_{|\alpha} + b_{\alpha\beta}(\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}K_{\rho\lambda}) = 0.
\end{aligned} \tag{40}$$

It is easy to note that between the homogeneous equilibrium equations (35) and the compatibility conditions (40) there exists the following correspondence:

$$\begin{aligned}
&N^{\alpha\beta} \Leftrightarrow \varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}K_{\rho\lambda}, \quad && Q^{\alpha} \Leftrightarrow -\varepsilon^{\alpha\rho}K_{\rho}, \\
&M^{\alpha\beta} \Leftrightarrow -\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}E_{\rho\lambda}, \quad && M^{\alpha} \Leftrightarrow -\varepsilon^{\alpha\rho}E_{\rho}.
\end{aligned} \tag{41}$$

When the resultant 2D stress measures in (35) are replaced by the 2D strain measures according to (41), the homogeneous equilibrium equations (35) are converted *exactly* into the compatibility conditions (40). The correspondence (41) is *the extended static-geometric analogy* in the resultant six-field linear theory of shells. Within the classical three-field linear shell model of K-L type the analogy was first noted by Gol'denveiser [26], while for the five-field linear shell model of T-R type it was first noted by Pelekh and Lun' [27]. The correspondence (41) naturally extends the static-geometric analogy to the resultant six-field linear theory of shells.

The correspondence (41) allows one to formulate all relations of the six-field linear shell theory in an easily memorable operator form, which is convenient for further algorithmization of the six-field shell model towards numerical applications. Such operator representation of the classical linear shell models were discussed in detail in the books by Başar and Krätzig [6, 38].

The correspondence property (41) of the linear six-field shell theory allows one to introduce six stress functions $\bar{u}_\alpha, \bar{w}, \bar{\psi}_\alpha, \bar{\psi}$ by the relations

$$
\begin{aligned}
&N^{\alpha\beta} = N_*^{\alpha\beta} + Ehc\,\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\bar{K}_{\rho\lambda}\,, \quad Q^\alpha = Q_*^\alpha - Ehc\,\varepsilon^{\alpha\rho}\bar{K}_\rho\,, \\
&M^{\alpha\beta} = M_*^{\alpha\beta} - Ehc\,\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\bar{E}_{\rho\lambda}\,, \quad M^\alpha = M_*^\alpha - Ehc\,\varepsilon^{\alpha\rho}\bar{E}_\rho\,, \quad c = \frac{h}{\sqrt{12(1-\nu^2)}}\,.
\end{aligned}
\tag{42}
$$

Here $N_*^{\alpha\beta}, Q_*^\alpha, M_*^{\alpha\beta}, M_*^\alpha$ is some particular solution of the inhomogeneous equilibrium equations (35), and the expressions $\bar{K}_{\rho\lambda}, \bar{K}_\rho, \bar{E}_{\rho\lambda}, \bar{E}_\rho$ are similar to (30) and (31) only constructed from the corresponding stress functions $\bar{u}_\alpha, \bar{w}, \bar{\psi}_\alpha, \bar{\psi}$. It is easy to see that indeed the relations (42) satisfy the equilibrium equations (35) for any value of the stress functions because of the compatibility conditions (40).

Let us introduce the following complex stress resultants and stress couples:

$$
\begin{aligned}
&\tilde{N}^{\alpha\beta} = N^{\alpha\beta} - i\,Ehc\,\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}K_{\rho\lambda}\,, \quad \tilde{Q}^\alpha = Q^\alpha + i\,Ehc\,\varepsilon^{\alpha\rho}K_\rho\,, \\
&\tilde{M}^{\alpha\beta} = M^{\alpha\beta} + i\,Ehc\,\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}E_{\rho\lambda}\,, \quad \tilde{M}^\alpha = M^\alpha + i\,Ehc\,\varepsilon^{\alpha\rho}E_\rho\,, \quad i = \sqrt{-1}\,.
\end{aligned}
\tag{43}
$$

Changing here the stress resultants and stress couples by their expressions (42), we obtain

$$
\begin{aligned}
&\tilde{N}^{\alpha\beta} = N_*^{\alpha\beta} - i\,Ehc\,\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\tilde{K}_{\rho\lambda}\,, \quad \tilde{Q}^\alpha = Q_*^\alpha + i\,Ehc\,\varepsilon^{\alpha\rho}\tilde{K}_\rho\,, \\
&\tilde{M}^{\alpha\beta} = M_*^{\alpha\beta} + i\,Ehc\,\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\tilde{E}_{\rho\lambda}\,, \quad \tilde{M}^\alpha = M_*^\alpha + i\,Ehc\,\varepsilon^{\alpha\rho}\tilde{E}_\rho\,,
\end{aligned}
\tag{44}
$$

where $\tilde{K}_{\rho\lambda}, \tilde{K}_\rho, \tilde{E}_{\rho\lambda}, \tilde{E}_\rho$ are now the expressions corresponding to (30) and (31), but constructed from the complex translations and rotations

$$
\tilde{u}_\alpha = u_\alpha + i\bar{u}_\alpha\,, \quad \tilde{w} = w + i\,\bar{w}\,, \quad \tilde{\psi}_\alpha = \psi_\alpha + i\bar{\psi}_\alpha\,, \quad \tilde{\psi} = \psi + i\bar{\psi}\,. \tag{45}
$$

When the system of equations (40) is multiplied by $-i\,Ehc$ and added with the corresponding equilibrium equations (35), this leads to the system of linearized equilibrium equations for the complex 2D stress measures,

$$
\begin{aligned}
&\tilde{N}^{\alpha\beta}|_\alpha - b_\alpha^\beta\tilde{Q}^\alpha + f^\beta = 0\,, \quad \tilde{Q}^\alpha|_\alpha + b_{\alpha\beta}\tilde{N}^{\alpha\beta} + f = 0\,, \\
&\tilde{M}^{\alpha\beta}|_\alpha - \tilde{Q}^\beta + \varepsilon^{\lambda\beta}b_{\alpha\lambda}\tilde{M}^\alpha + m^\beta = 0\,, \quad \tilde{M}^\alpha|_\alpha + \varepsilon_{\alpha\beta}(\tilde{N}^{\alpha\beta} - b_\lambda^\alpha\tilde{M}^{\lambda\beta}) + m = 0\,.
\end{aligned}
\tag{46}
$$

When expressed in terms of complex displacements (47), the above system of PDEs for the complex independent variables is of 6$^{\text{th}}$ order, while the system (35) of PDEs for the real displacements is of 12$^{\text{th}}$ order.

Corresponding dynamic and kinematic boundary conditions associated with the complex shell equations (46) are

$$
\begin{aligned}
&\mathrm{Re}\,\tilde{\boldsymbol{n}}^\alpha\nu_\alpha = \boldsymbol{n}^*\,, \quad \mathrm{Re}\,\tilde{\boldsymbol{m}}^\alpha\nu_\alpha = \boldsymbol{m}^* \quad \text{along } \partial M_f\,, \\
&\mathrm{Re}\,\tilde{\boldsymbol{u}} = \boldsymbol{u}^*\,, \quad \mathrm{Re}\,\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi}^* \quad \text{along } \partial M_d = \partial M \setminus \partial M_f\,.
\end{aligned}
\tag{47}
$$

The complex formulation of shell relations for the linear K-L theory of thin elastic shells was proposed by Novozhilov [28], generalizing a similar approach to the axisymmetric deformation of shells of revolution proposed by Meissner [29]. The complex BVP was subsequently used to solve a number of non-trivial shell problems, many of which were included in the books by Novozhilov [30], Chernykh [31] and Novozhilov et al. [7]. For the linear theory of elastic shells of T-R type the complex equilibrium equations were proposed by Pelekh and Lun' [27], and some analytical solutions of shell problems were presented in Pelekh [10]. Now one can use the complex BVP of the linear six-field shell theory with hope to obtain more accurate analytical results than those based on K-L and T-R type shell models.

5 Deformation of the shell lateral boundary element

In the resultant shell theory, the undeformed rectilinear shell lateral boundary surface $\partial \mathrm{B}^*$ deforms into the deformed lateral boundary surface $\chi(\partial \mathrm{B}^*)$, which is not the rectilinear one anymore, see Figs. 2 and 3. To describe the linearized deformation of $\partial \mathrm{B}^*$ into $\chi(\partial \mathrm{B}^*)$, let the boundary contour ∂M of the shell midsurface be parametrized by the length coordinate s. Along ∂M we can introduce the triad of orthonormal vectors: the tangent $\boldsymbol{\tau} = d\boldsymbol{x}/ds$, the $\boldsymbol{n}$ normal to M, and the exterior normal $\boldsymbol{\nu} = \boldsymbol{\tau} \times \boldsymbol{n}$. In the base $\boldsymbol{\nu}, \boldsymbol{\tau}, \boldsymbol{n}$ the translation $\boldsymbol{u}$ and rotation $\boldsymbol{\psi}$ vectors are represented by

$$
\begin{aligned}
&\boldsymbol{u} = u_\nu \boldsymbol{\nu} + u_\tau \boldsymbol{\tau} + w\boldsymbol{n}, \quad \boldsymbol{\psi} = -\psi_\tau \boldsymbol{\nu} + \psi_\nu \boldsymbol{\tau} + \psi \boldsymbol{n}, \\
&u_\nu = u_\alpha \nu^\alpha, \quad u_\tau = u_\alpha \tau^\alpha, \quad \psi_\nu = \psi_\alpha \nu^\alpha, \quad \psi_\tau = \psi_\alpha \tau^\alpha, \\
&\nu^\alpha = \boldsymbol{\nu} \cdot \boldsymbol{a}^\alpha, \quad \tau^\alpha = \boldsymbol{\tau} \cdot \boldsymbol{a}^\alpha, \quad \nu^\alpha = \varepsilon^{\alpha\beta} \tau_\beta, \quad \tau^\alpha = \varepsilon^{\beta\alpha} \nu_\beta .
\end{aligned}
\tag{48}
$$

With the shell deformation described by the infinitesimal $\boldsymbol{u}$ and $\boldsymbol{\psi}$ the orthonormal triad $\boldsymbol{\nu}, \boldsymbol{\tau}, \boldsymbol{n}$ is moved into the orthonormal triad $\boldsymbol{t}_\nu, \boldsymbol{t}_\tau, \boldsymbol{t}$ defined by

$$
\begin{aligned}
\boldsymbol{t}_\nu &= \boldsymbol{\nu} + \boldsymbol{\psi} \times \boldsymbol{\nu} = \boldsymbol{\nu} + \psi \boldsymbol{\tau} - \psi_\nu \boldsymbol{n}, \\
\boldsymbol{t}_\tau &= \boldsymbol{\tau} + \boldsymbol{\psi} \times \boldsymbol{\tau} = -\psi \boldsymbol{\nu} + \boldsymbol{\tau} - \psi_\tau \boldsymbol{n}, \\
\boldsymbol{t} &= \boldsymbol{n} + \boldsymbol{\psi} \times \boldsymbol{n} = \psi_\nu \boldsymbol{\nu} + \psi_\tau \boldsymbol{\tau} + \boldsymbol{n} .
\end{aligned}
\tag{49}
$$

Because $\boldsymbol{\psi}$ describes the global averaged rotation of neighbourhood of the boundary point $x \in \partial M$, there is no reason to expect that so defined $\boldsymbol{t}_\tau$ becomes tangent to the deformed boundary contour $\chi(\partial M)$.

According to the modified polar decomposition (18) the material fibre $\boldsymbol{\tau} ds$ tangent to ∂M should be rotated by a total rotation vector $\boldsymbol{\psi}_\tau$ into the fibre $\hat{\boldsymbol{\tau}} ds$ tangent to $\chi(\partial M)$, where $\hat{\boldsymbol{\tau}}$ denotes the unit vector tangent to the deformed boundary contour in $y \in \chi(\partial M)$. This can be achieved by two rotations: the global rotation $\boldsymbol{\psi}$ associated with the rotation tensor $\boldsymbol{Q}$ of the neighbourhood of $x \in \partial M$ followed by an additional rotation $\boldsymbol{\psi}_a$ associated with the stretch $\boldsymbol{\Lambda}$. Let us construct such a total rotation vector of the boundary contour appropriate for the linear six-field shell theory.

Since $\boldsymbol{y}(s) = \chi\left(\boldsymbol{x}(s)\right)$, the vector $d\boldsymbol{y}/ds$ becomes tangent to $\chi(\partial M)$ at $\boldsymbol{y} = \chi(\boldsymbol{x})$,

$$
\begin{aligned}
\frac{d\boldsymbol{y}}{ds} &= \frac{d\boldsymbol{x}}{ds} + \frac{d\boldsymbol{u}}{ds} = \boldsymbol{\tau} + \boldsymbol{u},_\alpha \tau^\alpha = \boldsymbol{\tau} + \boldsymbol{\varepsilon}_\alpha \tau^\alpha + \boldsymbol{\psi} \times \boldsymbol{\tau}, \\
&= (E_{\tau\nu} - \psi)\boldsymbol{\nu} + (1 + E_{\tau\tau})\boldsymbol{\tau} + (E_\tau - \psi_\tau)\boldsymbol{n},
\end{aligned}
\tag{50}
$$

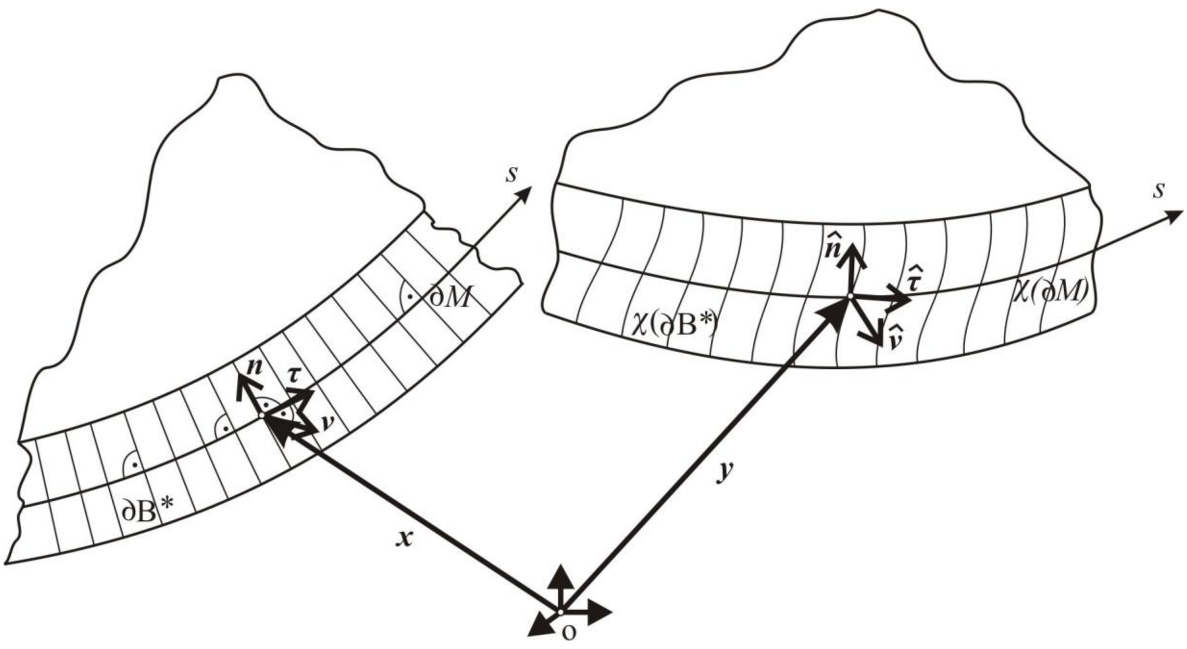

Figure 3. Deformation of the shell lateral boundary surface

where

$$
\begin{aligned}
E_{\tau\nu} - \psi &= E_{\alpha\beta}\tau^\alpha \nu^\beta - \psi = \frac{du_\nu}{ds} - \rho_\tau u_\tau + \tau_\tau w\,, \\
E_{\tau\tau} &= E_{\alpha\beta}\tau^\alpha \tau^\beta = \frac{du_\tau}{ds} + \rho_\tau u_\nu - \sigma_\tau w\,, \\
E_\tau - \psi_\tau &= (E_\alpha - \psi_\alpha)\tau^\alpha = \frac{dw}{ds} + \sigma_\tau u_\tau - \tau_\tau u_\nu\,,
\end{aligned}
\tag{51}
$$

and $\sigma_\tau = b_{\alpha\beta}\tau^\alpha\tau^\beta$ is the normal curvature, $\tau_\tau = -b_{\alpha\beta}\nu^\alpha\tau^\beta$ is the geodesic torsion, and $\rho_\tau = \tau_\alpha \nu^\alpha|_\beta \tau^\beta = -\nu_\alpha \tau^\alpha|_\beta \tau^\beta$ is the geodesic curvature of ∂M.
Now the length of the vector $\boldsymbol{dy}/ds$ following from (50) is

$$
\sqrt{\left(\frac{d\boldsymbol{y}}{ds}\right)^2} = \sqrt{(E_{\tau\nu} - \psi)^2 + (1 + E_{\tau\tau})^2 + (E_\tau - \psi_\tau)^2} \approx 1 + E_{\tau\tau}\,. \tag{52}
$$

The unit vector $\hat{\boldsymbol{\tau}}$ tangent to $\chi(\partial M)$ in $y = \chi(x)$ can be defined by

$$
\hat{\boldsymbol{\tau}} \approx \frac{d\boldsymbol{y}/ds}{1 + E_{\tau\tau}} \approx (E_{\tau\nu} - \psi)\boldsymbol{\nu} + \boldsymbol{\tau} + (E_\tau - \psi_\tau)\boldsymbol{n}\,. \tag{53}
$$

Please notice that $\hat{\boldsymbol{\tau}}$ does not coincide with $\boldsymbol{t}_\tau$ introduced by $(49)_2$.

In order to properly describe the linearized rotation of the rectilinear boundary element $\Delta s \times h$ defined by the triad $\boldsymbol{\nu}, \boldsymbol{\tau}, \boldsymbol{n}$, one has to introduce in the deformed placement an orthonormal triad containing $\hat{\boldsymbol{\tau}}$ as the unit tangent to $\chi(\partial M)$. This can be achieved by introducing two other unit vectors

$$
\begin{aligned}
\hat{\boldsymbol{\nu}} &= \hat{\boldsymbol{\tau}} \times \boldsymbol{t} = \boldsymbol{\nu} - (E_{\tau\nu} - \psi)\boldsymbol{\tau} - \psi_\nu \boldsymbol{n}\,, \\
\hat{\boldsymbol{n}} &= \hat{\boldsymbol{\nu}} \times \hat{\boldsymbol{\tau}} = \psi_\nu \boldsymbol{\nu} - (E_\tau - \psi_\tau)\boldsymbol{\tau} + \boldsymbol{n}\,.
\end{aligned}
\tag{54}
$$

The same unit vectors follow if one defines first $\hat{\boldsymbol{n}} = \boldsymbol{t}_\nu \times \hat{\boldsymbol{\tau}}$ and then $\hat{\boldsymbol{\nu}} = \hat{\boldsymbol{\tau}} \times \hat{\boldsymbol{n}}$. The triad $\hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\tau}}, \hat{\boldsymbol{n}}$ is orthonormal within the linear terms in infinitesimal translations and rotations. But here $\hat{\boldsymbol{n}}$ is neither normal to $\chi(M)$ nor tangent to $\chi(\partial\mathrm{B}^*)$ and $\hat{\boldsymbol{\nu}}$ is neither tangent to $\chi(M)$ nor normal to $\chi(\partial\mathrm{B}^*)$, because the deformed boundary surface $\chi(\partial\mathrm{B}^*)$ itself is not the rectilinear one, in general.

The triad $\hat{\boldsymbol{\nu}},\hat{\boldsymbol{\tau}},\hat{\boldsymbol{n}}$ allows one to introduce the total rotation vector $\boldsymbol{\psi}_\tau$ of the normal element $\Delta s\times h$ associated with the boundary contour ∂M by

$$\begin{aligned}\boldsymbol{\psi}_\tau &= \frac{1}{2}\left(\boldsymbol{\nu}\times\hat{\boldsymbol{\nu}}+\boldsymbol{\tau}\times\hat{\boldsymbol{\tau}}+\boldsymbol{n}\times\hat{\boldsymbol{n}}\right)=(E_\tau-\psi_\tau)\boldsymbol{\nu}+\psi_\nu\boldsymbol{\tau}-(E_{\tau\nu}-\psi)\boldsymbol{n}\\ &=\boldsymbol{\psi}+\boldsymbol{\psi}_a\,,\quad \boldsymbol{\psi}_a=E_\tau\boldsymbol{\nu}-E_{\tau\nu}\boldsymbol{n}\,.\end{aligned}\tag{55}$$

The physical meaning of the additional rotation vector $\boldsymbol{\psi}_a$ is apparent: it allows one to bring the triad $\boldsymbol{\nu},\boldsymbol{\tau},\boldsymbol{n}$, initially rotated into $\boldsymbol{t}_\nu,\boldsymbol{t}_\tau,\boldsymbol{t}$ by the global linearized rotation vector $\boldsymbol{\psi}$, to coincide with the triad $\hat{\boldsymbol{\nu}},\hat{\boldsymbol{\tau}},\hat{\boldsymbol{n}}$.

With $\boldsymbol{\psi}_\tau$ defined in (55), differentiation of the translation field along ∂M gives

$$\frac{d\boldsymbol{u}}{ds}=E_{\tau\tau}\boldsymbol{\tau}+\boldsymbol{\psi}_\tau\times\boldsymbol{\tau}\,,\tag{56}$$

where the right-hand side is expressible by three stretches $E_{\tau\nu},E_{\tau\tau},E_\tau$ and three linearized rotations ψ_ν,ψ_τ,ψ along ∂M.

Warping properties of the boundary element $\Delta s\times h$ can be characterized by the vector of change of boundary curvature defined by

$$\boldsymbol{k}_\tau=\frac{d\boldsymbol{\psi}_\tau}{ds}=-k_{\tau\tau}\boldsymbol{\nu}+k_{\tau\nu}\boldsymbol{\tau}+k_\tau\boldsymbol{n}\,,\tag{57}$$

$$k_{\tau\tau}=K_{\tau\tau}-\frac{dE_\tau}{ds}+\tau_\tau E_{\tau\nu}\,,\quad k_{\tau\nu}=K_{\tau\nu}+\sigma_\tau E_{\tau\nu}+\rho_\tau E_\tau\,,\quad k_\tau=K_\tau-\frac{dE_{\tau\nu}}{ds}-\tau_\tau E_\tau\,,\tag{58}$$

where

$$\begin{aligned}K_{\tau\tau}&=K_{\alpha\beta}\tau^\alpha\tau^\beta=\frac{d\psi_\tau}{ds}+\rho_\tau\psi_\nu-\tau_\tau\psi\,,\\ K_{\tau\nu}&=K_{\alpha\beta}\tau^\alpha\nu^\beta=\frac{d\psi_\nu}{ds}-\rho_\tau\psi_\tau-\sigma_\tau\psi\,,\\ K_\tau&=K_\alpha\tau^\alpha=\frac{d\psi}{ds}+\sigma_\tau\psi_\nu+\tau_\tau\psi_\tau\,.\end{aligned}\tag{59}$$

The vector $\boldsymbol{k}_\tau$ is expressed by (57) entirely through the 2D surface strain measures $E_{\tau\nu},E_{\tau\tau},E_\tau$ and $K_{\tau\nu},K_{\tau\tau},K_\tau$.

In analogy to [31], the relation (57) can be integrated along ∂M to obtain

$$\boldsymbol{\psi}_\tau=\boldsymbol{\psi}_\tau^0+\int_{s_0}^{s}\boldsymbol{k}_\tau ds'\,.\tag{60}$$

From (56) it also follows that

$$\boldsymbol{u}=\boldsymbol{u}^0+\int_{s_0}^{s}\left(E_{\tau\tau}\boldsymbol{\tau}'+\boldsymbol{\psi}_\tau\times\boldsymbol{\tau}'\right)ds'\,.\tag{61}$$

Since $\boldsymbol{\tau}=d(\boldsymbol{x}-\boldsymbol{x}_0)/ds$, we can integrate by parts the last term in (61) leading to

$$\int_{s_0}^{s}\boldsymbol{\psi}_\tau\times\boldsymbol{\tau}'ds'=\boldsymbol{\psi}_\tau\times(\boldsymbol{x}-\boldsymbol{x}_0)-\int_{s_0}^{s}\boldsymbol{k}_\tau\times(\boldsymbol{x}'-\boldsymbol{x}_0)\,ds'\,,\tag{62}$$

which together with (60) allows one to present (61) in the form

$$\boldsymbol{u}=\boldsymbol{u}^0+\boldsymbol{\psi}_\tau^0\times(\boldsymbol{x}-\boldsymbol{x}_0)+\left(\int_{s_0}^{s}\boldsymbol{k}_\tau\,ds'\right)\times(\boldsymbol{x}-\boldsymbol{x}_0)+\int_{s_0}^{s}\left[E_{\tau\tau}\boldsymbol{\tau}'+(\boldsymbol{x}'-\boldsymbol{x}_0)\times\boldsymbol{k}_\tau\right]ds'\,.\tag{63}$$

But using the identity

$$\frac{d}{ds}\left\{\left(\int_{s_0}^{s}\boldsymbol{k}_\tau ds'\right)\times(\boldsymbol{x}-\boldsymbol{x}_0)-\int_{s_0}^{s}\left[\boldsymbol{k}_\tau\times(\boldsymbol{x}'-\boldsymbol{x}_0)\right]ds'\right\}=\left(\int_{s_0}^{s}\boldsymbol{k}_\tau ds'\right)\times\boldsymbol{\tau}\,, \tag{64}$$

the relation (63) can be transformed into the more concise form

$$\boldsymbol{u}=\boldsymbol{u}^0+\boldsymbol{\psi}_\tau^0\times(\boldsymbol{x}-\boldsymbol{x}_0)+\int_{s_0}^{s}\left[\left(\int_{s_0}^{s'}\boldsymbol{k}_\tau ds''\right)\times\boldsymbol{\tau}'+E_{\tau\tau}\boldsymbol{\tau}'\right]ds'\,. \tag{65}$$

The relation (65), derived here along the boundary contour ∂M, is valid also for any regular curve C on the shell base surface. In case of the linear K-L type shell theory, it was indicated in [31] that for the simply connected base surface M the resulting translations (61) and rotations (60) do not depend on the type of surface curve connecting the initial x_0 and current x points on M. Thus, the relations (60) and (65) can be treated as the Cesáro type formulas in the resultant linear six-field theory of shells.

The results given in sections 3 and 4 generalize those known in the literature for the linear K-L and T-R type shell models. However, a direct reduction of our results to those of the simpler shell models needs some reinterpretation of the basic fields of shell theory. Our independent kinematic field variables $\boldsymbol{u},\boldsymbol{\psi}$ associated with M describe the energetic through-the-thickness averaged gross displacement of the shell cross section. A similar meaning of through-the-thickness energetic averages have our 2D surface strain vectors $\boldsymbol{\varepsilon}_\alpha,\boldsymbol{\kappa}_\alpha$ associated with M. In the classical linear shell models, the 2D displacement and strain fields are defined approximately from corresponding 3D fields either by enforcing some kinematic constraints or by expanding the corresponding 3D fields into polynomials in the normal direction and defining the shell kinematic fields as the zeroth- and/or first-order terms of the expansion. Such geometrically defined 2D fields have different physical meaning than our energetically defined fields used here for the resultant linear six-field theory of shells.

For example, in the classical linear K-L shell model, developed in many papers and books, for description of shell kinematics usually the kinematic constraint “the material fibres, which are normal to the undeformed shell base surface, remain normal to the deformed base surface and do not change their lengths” is used. Under this constraint, the 2D translation vector $\boldsymbol{u}$ means just the translation of M into $N=\chi(M)$, while the linearized rotation vector is not an independent field variable, but is expressed through components of $\boldsymbol{u}$ and their surface derivatives. Both shell strain measures become the symmetric tensors defined only by $\boldsymbol{u}$ and its surface gradients. Nevertheless, if one simplifies all the fields of the linear six-field shell model given here under such kinematic constraint one can interpret them as being approximately equivalent to corresponding fields of the K-L type shell model given in the book by Novozhilov et. al. [7], Chapter VIII. In particular, under this interpretation our formulas (56), (57), and (65) reduce to the corresponding formulas given in that book.

In the T-R type linear shell model, the relaxed kinematic constraint “the material fibres, which are normal to the undeformed shell base surface, remain straight after deformation process and do not change their lengths” is used. In this case, the translation vector of M and two linearized rotations about tangents to M are the independent field variables. We may again reduce our six-field shell relations under this constraint and interpret them as being approximately equivalent to the corresponding relations given for example in the books [8, 21]. In particular, under such interpretation our results for linearized deformation of the shell boundary element can be shown to be compatible with those following from linearization of corresponding geometrically non-linear results of Pietraszkiewicz [32].

When solving shell problems in terms of stress functions it is of importance to know how the stress function vectors $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{\psi}}$ are related to the resulting force and moment vectors of loads acting along the boundary contour ∂M_f.

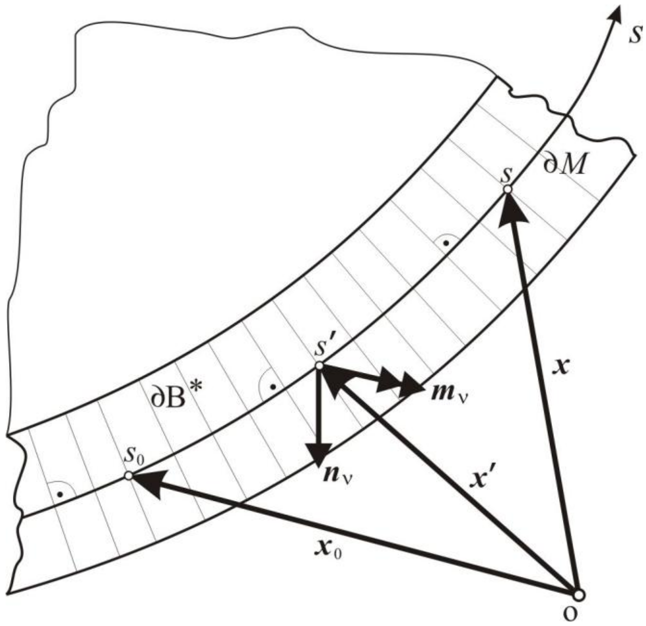

Figure 4. The resulting forces and couples along the shell boundary

According to (4) and (42), the boundary loads $\boldsymbol{n}^{\alpha}\nu_{\alpha}$ and $\boldsymbol{m}^{\alpha}\nu_{\alpha}$ per unit length of ∂M_f are

$$\begin{aligned}\boldsymbol{n}^{\alpha}\nu_{\alpha} &= \left(N^{*}_{\nu\nu}\boldsymbol{\nu} + N^{*}_{\nu\tau}\boldsymbol{\tau} + Q^{*}_{\nu}\boldsymbol{n}\right) - Ehc\left(-\bar{K}_{\tau\tau}\boldsymbol{\nu} + \bar{K}_{\tau\nu}\boldsymbol{\tau} + \bar{K}_{\tau}\boldsymbol{n}\right) = \boldsymbol{n}^{*}_{\nu} - Ehc\,\bar{\boldsymbol{\kappa}}_{\tau}\ , \\ \boldsymbol{m}^{\alpha}\nu_{\alpha} &= \left(-M^{*}_{\nu\tau}\boldsymbol{\nu} + M^{*}_{\nu\nu}\boldsymbol{\tau} + M^{*}_{\nu}\boldsymbol{n}\right) - Ehc\left(-\bar{E}_{\tau\nu}\boldsymbol{\nu} + \bar{E}_{\tau\tau}\boldsymbol{\tau} + \bar{E}_{\tau}\boldsymbol{n}\right) = \boldsymbol{m}^{*}_{\nu} - Ehc\left(\bar{E}_{\tau\tau}\boldsymbol{\tau} + \bar{\boldsymbol{\psi}}_{a}\times\boldsymbol{\tau}\right).\end{aligned} \tag{66}$$

The resulting force vector $\mathbf{F}$ of the loads $\boldsymbol{n}_{\nu}$ acting along the part $s_0 s$ of ∂M_f is

$$\mathbf{F} = \int_{s_0}^{s} \boldsymbol{n}_{\nu}ds' = \int_{s_0}^{s}\left(\boldsymbol{n}^{*}_{\nu} - Ehc\,\bar{\boldsymbol{\kappa}}_{\tau}\right)ds'. \tag{67}$$

Applying the relation $\bar{\boldsymbol{\kappa}}_{\tau} = d\bar{\boldsymbol{\psi}}/ds$ following from $(28)_2$ and (42), we can integrate (67) along ∂M_f to obtain

$$\mathbf{F} = \mathbf{F}^{*} - Ehc\left(\bar{\boldsymbol{\psi}} - \bar{\boldsymbol{\psi}}^{0}\right), \quad \mathbf{F}^{*} = \int_{s_0}^{s}\boldsymbol{n}^{*}_{\nu}ds', \tag{68}$$

where $\mathbf{F}^{*}$ is the resulting force vector corresponding to the chosen particular solution of the inhomogeneous equilibrium equations (35).

The resulting moment vector $\mathbf{B}_{(\mathrm{o})}$ taken with regard to a spatial origin $\mathrm{o}\in\mathcal{E}$ of the loads $\boldsymbol{n}_{\nu}$ and $\boldsymbol{m}_{\nu}$ acting along the part $s_0 s$ of ∂M_f is (see Fig. 4)

$$\begin{aligned}\mathbf{B}_{(\mathrm{o})} &= \int_{s_0}^{s}\left(\boldsymbol{m}_{\nu} + \boldsymbol{x}'\times\boldsymbol{n}_{\nu}\right)ds' \\ &= \int_{s_0}^{s}\left(\boldsymbol{m}^{*}_{\nu} + \boldsymbol{x}'\times\boldsymbol{n}^{*}_{\nu}\right)ds' - Ehc\int_{s_0}^{s}\left(\bar{E}_{\tau\tau}\boldsymbol{\tau}' + \bar{\boldsymbol{\psi}}_{a}\times\boldsymbol{\tau}' + \boldsymbol{x}'\times\bar{\boldsymbol{\kappa}}_{\tau}\right)ds'.\end{aligned} \tag{69}$$

But since in analogy to (56) and (55),

$$\bar{E}_{\tau\tau}\boldsymbol{\tau}+\bar{\boldsymbol{\psi}}_a\times\boldsymbol{\tau}=\frac{d\bar{\boldsymbol{u}}}{ds}-\bar{\boldsymbol{\psi}}\times\boldsymbol{\tau}\,, \tag{70}$$

the last integral of (69) can be integrated by parts,

$$\int_{s_0}^{s}\left(\frac{d\bar{\boldsymbol{u}}}{ds'}-\bar{\boldsymbol{\psi}}\times\boldsymbol{\tau}'+\boldsymbol{x}'\times\frac{d\bar{\boldsymbol{\psi}}}{ds'}\right)ds'=\bar{\boldsymbol{u}}-\bar{\boldsymbol{u}}^0+\boldsymbol{x}\times\bar{\boldsymbol{\psi}}-\boldsymbol{x}_0\times\bar{\boldsymbol{\psi}}^0\,, \tag{71}$$

so that

$$\mathbf{B}_{(\mathrm{o})}=\int_{s_0}^{s}\left(\boldsymbol{m}_\nu^*+\boldsymbol{x}'\times\boldsymbol{n}_\nu^*\right)ds'-Ehc\left(\bar{\boldsymbol{u}}-\bar{\boldsymbol{u}}^0+\boldsymbol{x}\times\bar{\boldsymbol{\psi}}-\boldsymbol{x}_0\times\bar{\boldsymbol{\psi}}^0\right). \tag{72}$$

The resulting moment vector $\mathbf{B}$ of the loads $\boldsymbol{n}_\nu$ and $\boldsymbol{m}_\nu$ taken with regard to the current point $x\in\partial M_f$ with the coordinate s is

$$\mathbf{B}=\mathbf{B}_{(\mathrm{o})}-\boldsymbol{x}\times\mathbf{F}=\mathbf{B}^*-Ehc\left[\bar{\boldsymbol{u}}-\bar{\boldsymbol{u}}^0+(\boldsymbol{x}-\boldsymbol{x}_0)\times\bar{\boldsymbol{\psi}}^0\right], \tag{73}$$

where now

$$\mathbf{B}^*=\int_{s_0}^{s}\left[\boldsymbol{m}_\nu^*+(\boldsymbol{x}'-\boldsymbol{x})\times\boldsymbol{n}_\nu^*\right]ds' \tag{74}$$

is the resulting moment vector with regard to a current point $x\in\partial M_f$ corresponding to the chosen particular solution of the inhomogeneous equilibrium equations (35).

Solving (68) and (73) for the stress function vectors $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{\psi}}$ we obtain

$$\bar{\boldsymbol{\psi}}=\bar{\boldsymbol{\psi}}^0-\frac{\mathbf{F}-\mathbf{F}^*}{Ehc}\,,\quad \bar{\boldsymbol{u}}=\bar{\boldsymbol{u}}^0+\bar{\boldsymbol{\psi}}^0\times(\boldsymbol{x}-\boldsymbol{x}_0)-\frac{\mathbf{B}-\mathbf{B}^*}{Ehc}\,. \tag{75}$$

The relations (75) along the boundary contour ∂M_f are also valid for any regular curve C on the shell base surface M. In the case of simply connected M and the closed curve C we can assume $\bar{\boldsymbol{u}}^0=\bar{\boldsymbol{\psi}}^0=\mathbf{0}$. This means the change of stress function vectors by the term $\bar{\boldsymbol{u}}^0+\bar{\boldsymbol{\psi}}^0\times(\boldsymbol{x}-\boldsymbol{x}_0)$ of “rigid-body motion” type which does not influence the stress-deformation state of the shell. In the case of shell problems with a multi-connected M the relations (75) should allow to construct multivalued parts of the stress functions in analogy to procedures discussed for the linear K-L type elastic shells developed in [31, 33, 7].

6 Gradients of 2D shell measures in the resultant stress working

For reasons noted in Introduction, Pietraszkiewicz [16] had to introduce to the resultant non-linear 2D energy balance an additional 2D mechanical power called an interstitial working. Only then the so refined 2D balance of energy and the 2D entropy inequality could be regarded as the exact resultant implications of corresponding 3D laws of rational thermomechanics. But then the constitutive equations even of thermoelastic shells were allowed to depend also on surface gradients of the 2D strain measures, in general.

We are not aware of any discussion of possible appearance of the strain gradients in 2D constitutive equations of linearly elastic shells following from a 3D-to-2D reduction of non-polar elastic solids. To have some insight into the problem, let us analyse the 3D-to-2D reduction of the exact 3D stress power density (22) for the special case of the resultant linear six-field theory of isotropic elastic shells. In this case, time derivatives of various fields become increments of the fields from the undeformed shell state, and the 2D stress power reduces to the 2D stress working. In equilibrium problems, the first two terms in (22), but without overdots, represent the effective 2D stress working expressed entirely in terms of 2D shell stress and strain measures. Any possible non-classical terms may only be hidden in the last three integrals of (22) containing the intrinsic deviation vector $\mathbf{e}(\theta^{\alpha},\xi)$.

To estimate the components of $\mathbf{e}$ we need to know quadratic and cubic parts of 3D displacement distributions within the shell space. Unfortunately, for the resultant six-field linear shell theory the discussion of possible quadratic and cubic parts of displacement and stress distributions across the thickness is not available in the literature yet. But for our purposes it is sufficient to estimate only orders of magnitude of such terms, not to have their explicit exact expressions. Thus, in this paper we shall use the approximate results derived by Rychter [34] for the Reissner type linear theory of isotropic elastic shells: his Eqs. (26) and (27) for the kinematically admissible components of 3D displacement field, and his Eqs. (30) for the statically admissible components of 3D stress field. In our notation and conventions, the components of $\mathbf{e}(\theta^{\alpha},\xi)$ following from the 3D displacement distributions of [34] are:

$$e_{\rho} \approx k(\xi)q_{\rho} + g(\xi)c_{\rho}\,, \quad e \approx r(\xi)q + s(\xi)d\,, \tag{76}$$

where

$$k(\xi)=\frac{4\xi^2}{h^2}-\frac{1}{3}, \quad g(\xi)=\frac{8\xi^3}{h^3}-\frac{6\xi}{5h}, \quad r(\xi)=\frac{20\xi^2}{h^2}-1, \quad s(\xi)=\frac{8\xi^3}{h^3}-\frac{2\xi}{h}, \tag{77}$$

$$\begin{gathered} q_{\rho}=\frac{h}{4}D^{\lambda\mu}E_{(\lambda\mu)},_{\rho}\ , \quad c_{\rho}=\frac{h^3}{48}D^{\lambda\mu}K_{(\lambda\mu)},_{\rho}-\frac{5}{24}hE_{\rho}\,, \\ q=-\frac{h^2}{40}D^{\lambda\mu}K_{(\lambda\mu)}\,, \quad d=-\frac{h^3}{48}D^{\alpha\beta}D^{\lambda\mu}E_{(\lambda\mu)|\alpha\beta}\,, \quad D^{\alpha\beta}=\frac{\nu}{1-\nu}a^{\alpha\beta}. \end{gathered} \tag{78}$$

In the statically admissible stress distributions needed in (22) it is sufficient for our purpose to take into account only the main terms (see [34], Eq. (30c), and [20], Eqs. (28)),

$$\begin{aligned} \mathbf{\Lambda s}^{\alpha}\mu &\approx \left(\frac{1}{h}N^{\alpha\beta}+\frac{12}{h^3}M^{\alpha\beta}\xi\right)\boldsymbol{a}_{\beta}+\frac{1}{h}Q^{\alpha}f(\xi)\boldsymbol{n}\,, \\ \mathbf{\Lambda s}^{3}\mu &\approx \frac{1}{h}Q^{\alpha}f(\xi)\boldsymbol{a}_{\alpha}+\left[-b_{\alpha\beta}N^{\alpha\beta}m(\xi)+\frac{1}{h}b_{\alpha\beta}M^{\alpha\beta}f(\xi)-M^{\alpha\beta}{}_{|\beta\alpha}l(\xi)\right]\boldsymbol{n}\,, \end{aligned} \tag{79}$$

$$m(\xi)=\frac{1}{2}+\frac{\xi}{h}, \quad f(\xi)=\frac{3}{2}\left(1-\frac{4\xi^2}{h^2}\right), \quad l(\xi)=\frac{1}{2}+\frac{3\xi}{2h}-\frac{2\xi^3}{h^3}. \tag{80}$$

Now one has to introduce the relations (77) - (80) into the last three terms of (22) without overdots, calculate explicitly all the non-vanishing integrals, and then select the leading terms with estimated highest orders. In such an elementary but involved through-the-thickness integration and estimation procedure, which we do not fully reproduce here for brevity of presentation, one has to account that only polynomials with highest terms containing ξ^n with $n=0,2,4,\ldots$ should be integrated. Integrals of polynomials with highest terms containing ξ^n with $n=1,3,5,\ldots$ vanish identically for our symmetric bounds of integration $\pm h/2$. One has also use some integration formulae given in [20], Eqs. (46), and the following formulae:

$$\int_-^+ f(\xi)k(\xi)d\xi = -\frac{2}{15}h\,,\quad \int_-^+ f(\xi)r(\xi)d\xi = \frac{3}{4}h\,,\quad \int_-^+ r(\xi)d\xi = \frac{2}{3}h\,,\quad \int_-^+ s(\xi)\xi d\xi = -\frac{h^2}{15}\,,$$

$$\int_-^+ g(\xi),_\xi d\xi = \frac{4}{5}\,,\quad \int_-^+ f(\xi)g(\xi),_\xi d\xi = 0\,,\quad \int_-^+ f(\xi)s(\xi),_\xi d\xi = -\frac{4}{5}\,,$$

$$\int_-^+ l(\xi)k(\xi),_\xi d\xi = \frac{9}{5}\,,\quad \int_-^+ l(\xi)r(\xi),_\xi d\xi = 9\,,\quad \int_-^+ m(\xi)k(\xi),_\xi d\xi = \frac{5}{3}\,,\quad \int_-^+ m(\xi)r(\xi),_\xi d\xi = \frac{25}{3}\,. \tag{81}$$

When we introduce the relations (76) - (81) into the last three integrals of (22) after appropriate transformations we obtain

$$\begin{aligned} &\int_-^+ (\mathbf{e}\times\mathbf{\Lambda s}^\alpha)\,\mu d\xi\cdot\boldsymbol{\kappa}_\alpha \approx \left(\frac{2}{15}Q^\alpha q^\beta + \frac{2}{3}N^{\alpha\beta}q - \frac{4}{5h}M^{\alpha\beta}d\right)K_{(\alpha\beta)}\,,\\ &\int_-^+ (\mathbf{\Lambda s}^\alpha)\cdot\mathbf{e},_\alpha\,\mu d\xi \approx \frac{3}{4}Q^\alpha q,_\alpha\,,\\ &\int_-^+ (\mathbf{\Lambda s}^3)\cdot\mathbf{e},_3\,\mu d\xi \approx -\frac{25}{3}b_{\alpha\beta}N^{\alpha\beta}q + \frac{1}{h}b_{\alpha\beta}M^{\alpha\beta}d - 9M^{\alpha\beta}{}_{|\beta\alpha}q\,. \end{aligned} \tag{82}$$

Let us now recall some order estimates derived by Koiter [35, 36] and John [37] for thin isotropic elastic shells undergoing small strains. In the case of vanishing surface and body forces the orders of some fields were estimated as

$$\begin{aligned} &a_{\alpha\beta} = O(1)\,,\quad b_{\alpha\beta} = O\left(\frac{1}{R}\right) = O\left(\frac{\theta^2}{h}\right),\\ &L = \min_{x\in M}\left(l, L_E, L_K\right),\quad \theta = \max_{x\in M}\left(\frac{h}{L}, \frac{h}{b}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\right),\quad \theta^2 << 1\,,\\ &N^{\alpha\beta} = O(Eh\eta)\,,\quad M^{\alpha\beta} = O(Eh^2\eta)\,,\quad Q^\alpha = O(Eh\eta\theta)\,,\\ &E_{\alpha\beta} = O(\eta)\,,\quad E_\alpha = O(\eta\theta)\,,\quad K_{\alpha\beta} = O\left(\frac{\eta}{h}\right), \end{aligned} \tag{83}$$

where $O(.)$ means “of the order of”, R is the smallest radius of curvature of M at $x\in M$, l is the characteristic length of geometric patterns of M, L_E and L_K are the characteristic lengths of extensional and bending patterns of shell deformation, η is the smallest stretch in the shell space, b is the distance of internal shell points to the shell lateral boundary surface ∂B^*, and θ is the small parameter.

To estimate orders of surface gradients of some fields we use the large parameter λ defined by

$$\lambda = \min_{x\in M}\left(L, b, \sqrt{hR}, \frac{h}{\sqrt{\eta}}\right),\quad (\cdot),_\alpha = O\left((\cdot)\frac{1}{\lambda}\right). \tag{84}$$

With (83) and (84) we have the following order estimates for the components (78) of $\mathbf{e}$:

$$q_\alpha = O\left(\frac{1}{8}\nu\eta\frac{h}{\lambda}\right),\quad q = O\left(\frac{1}{40}\nu h\eta\right),\quad d = O\left(\frac{1}{48}\nu^2 h\eta\frac{h^2}{\lambda^2}\right), \tag{85}$$

where numerical factors are still kept for further discussion.

Applying the relations (82) to (85), we are able to estimate all terms in the right-hand side of $(82)_1$,

$$\frac{2}{15}Q^{\alpha}q^{\beta}K_{(\alpha\beta)} = Eh\eta^2 \cdot O\left(\frac{1}{60}\nu\theta\frac{\eta}{\lambda}\right), \quad \frac{2}{3}N^{\alpha\beta}qK_{(\alpha\beta)} = Eh\eta^2 \cdot O\left(\frac{1}{60}\nu\eta\right),$$
$$\frac{4}{5h}M^{\alpha\beta}dK_{(\alpha\beta)} = Eh\eta^2 \cdot O\left(\frac{1}{60}\nu^2\eta\frac{h^2}{\lambda^2}\right), \tag{86}$$

so that the expression $(82)_1$ can be estimated by its largest second term as

$$\int_-^+ (\mathbf{e}\times\mathbf{\Lambda s}^{\alpha})\,\mu d\xi\cdot\boldsymbol{\kappa}_{\alpha} = Eh\eta^2 \cdot O\left(\frac{1}{60}\nu\eta\right). \tag{87}$$

The expression $(82)_2$ can be estimated by

$$\int_-^+ (\mathbf{\Lambda s}^{\alpha})\cdot\mathbf{e},_{\alpha}\,\mu d\xi = Eh\eta^2 \cdot O\left(\frac{3}{160}\nu\theta\frac{h}{\lambda}\right). \tag{88}$$

Similarly, all terms in the right-hand side of $(82)_3$ can be estimated as follows:

$$\frac{1}{h}b_{\alpha\beta}M^{\alpha\beta}d = Eh\eta^2 \cdot O\left(\frac{1}{48}\nu^2\theta^2\frac{h^2}{\lambda^2}\right), \quad -\frac{25}{3}b_{\alpha\beta}N^{\alpha\beta}q = Eh\eta^2 \cdot O\left(\frac{5}{24}\nu\theta^2\right),$$
$$-9M^{\alpha\beta}{}_{|\beta\alpha}q = Eh\eta^2 \cdot O\left(\frac{9}{40}\nu\frac{h^2}{\lambda^2}\right), \tag{89}$$

so that the expression $(82)_3$ can be estimated by its two first terms as

$$\int_-^+ (\mathbf{\Lambda s}^{3})\cdot\mathbf{e},_{3}\,\mu d\xi = Eh\eta^2 \cdot O\left(\max_{x\in M}(\nu\theta^2, \nu\frac{h^2}{\lambda^2})\right). \tag{90}$$

According to the estimates given in [20], within the small strain theory of isotropic elastic shells the first two terms of the resultant 2D stress working (22) consist of the main terms $N^{\alpha\beta}E_{\alpha\beta}$ and $M^{\alpha\beta}K_{\alpha\beta}$ which are $O(Eh\eta^2)$, of the secondary terms $Q^{\alpha}E_{\alpha}$ which are $=O(Eh\eta^2\theta^2)$, and of the terms $M^{\alpha}K_{\alpha}$ of smaller order. But when the constitutive equations (36) - (38) of the 2nd approximation theory are used, their secondary terms multiplied by corresponding $N^{\alpha\beta}$ and $M^{\alpha\beta}$ become $O(Eh\eta^2\theta^2)$ as well.

The additional terms in (22) containing the deviation vector $\mathbf{e}$ are calculated in (82) and their contributions to (22) are estimated in (87), (88), and (90) to be

$$Eh\eta^2\cdot\max_{x\in M} O\left(\frac{1}{60}\nu\eta, \frac{3}{160}\nu\theta\frac{h}{\lambda}, \nu\theta^2, \nu\frac{h^2}{\lambda^2}\right). \tag{91}$$

In the estimation procedure developed in [35-37] the approximate relation $\theta = O(h/\lambda)$ was used. Please also note that the first two terms in (91) contain small numerical factors which essentially reduce values of these terms as compared with the main terms in (22). As a result, the contribution to (22) of the main terms of (91) becomes $O(Eh\eta^2\nu\theta^2)$. Therefore, the contribution of the surface gradients of 2D shell strain measures to the resultant stress working are of the same order as of the secondary terms following from the effective part $\mathbf{n}^{\alpha}\cdot\boldsymbol{\varepsilon}_{\alpha} + \mathbf{m}^{\alpha}\cdot\boldsymbol{\kappa}_{\alpha}$. This indicates that while the surface gradients of 2D shell stress and strain measures do not contribute to the 1st approximation shell models, their contribution to the 2nd approximation to the complementary energy density is of the order of secondary terms of the energy. The questions of whether and/or how to consistently incorporate the gradients of 2D shell measures into the constitutive equations of the 2nd approximation linear six-field shell model should be addressed separately.

7 Conclusions

We have formulated the relations of the resultant linear six-field theory of elastic shells by consistent linearization of corresponding relations of the resultant non-linear theory of shells. Contrary to the classical linear three-field K-L type and five-field T-R type shell models, kinematics of the resultant linear six-field shell model is described by six independent components of the infinitesimal translation and rotation vectors of the shell base surface. Additionally, in the six-field linear shell model the resultant 2D stress couple vectors and corresponding 2D bending vectors all have the drilling components. The presence of these additional degrees of freedom is important for analyses of irregular shell structures containing junctions, self-intersections, shell-to-beam transitions, etc.

Within the resultant six-field linear theory of elastic shells we have formulated several results which are not available elsewhere. Among the new results let us point out the following:

- Formulation of the extended static-geometric analogy and derivation of complex shell relations for the complex displacements (section 4).
- Description of infinitesimal deformation of the shell boundary element (section 5) and derivation of the corresponding Cesáro type formulas (60) and (65).
- Expressions (75) for the vectors of stress functions in terms of the resulting force and moment vectors along an arbitrary curve on the shell base surface.
- Discussion on possible appearance of the surface gradients of 2D stress and strain measures in the resultant 2D stress working.

These theoretical results should be of interest to specialists of the linear theory of shells and those developing computer FEM software for analyses of irregular multi-shell structures.

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