On a description of deformable junction in the resultant non-linear shell theory

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Dedicated to Professor Holm Altenbach on His 60th birthday

Abstract The virtual work principle for two regular shell elements joined together along a part of their boundaries is proposed within the general non-linear resultant shell theory. It is assumed that translations across the junction curve are smooth, but no restrictions are enforced for the rotations. For stiff and hinge type junctions the curvilinear integral along the junction curve vanishes identically. In the case of deformable junction, the 1D constitutive type relation is proposed, where the constitutive function should be established by experiments for each particular engineering construction of the junction.

1. Introduction

By junctions of shells we mean design elements used for assembling regular shell parts along some of their boundaries into the complex multi-shell structure.

It follows from the review by Pietraszkiewicz and Konopińska [1] that different shell models available in the literature require special forms of jump conditions at the singular surface curves modelling the shell junctions. Jump conditions corresponding to different shell models may lead to different stress and strain distributions near the junction. But the review also indicates that almost in all descriptions of shell junctions available in the literature the stiff junction conditions were enforced. Deformability of the junction itself was explicitly indicated and used in only a few papers based on simplest shell models. This is in sharp contrast to the analyses and design of one-dimensional steel framed structures, where various semi-rigid beam-to-column connections were discussed in a number of papers, summarized in several books e.g. [2, 3] and were even introduced into Eurocode 3, [4].

Within the resultant non-linear six-field shell model, the mechanical theory of compound multi-shell structures was initiated by Makowski and Stumpf [5] and developed in the book by Chróścielewski et al. [6]. In this approach several regular shell elements may be joined at the common junction, deformability of any of the shell branches at the junction may individually be defined, and the junction curve itself may be equipped with additional mechanical properties independent from the adjacent shell branches. Unfortunately, the BVP of such a general theory became extremely complex and virtually useless for engineering applications. Even relatively simple cases of branching and self-intersecting shells developed in [7, 8] led to complex shell relations which were still hardly readable for engineering community. This explains why in the review [1] only a few papers on compound shell structures with deformable junctions was noted.

In this paper we formulate the variational principle of virtual work for the simple compound shell structure under the following assumptions:

- The structure consists of only two regular shell elements joined along a part of their boundaries.
- Translations of the whole base surface, including the junction curve, are smooth.

By further constraining the junction behaviour the stiff junction, the hinge junction, and the deformable junction are described and the corresponding reduced forms of the PVW are derived.

2 Notation and basic shell equilibrium conditions

The system of notations used here and basic shell relations are compatible with the ones used in the book by Chróścielewski et al. [6] and papers by Konopińska and Pietraszkiewicz [7], and Pietraszkiewicz and Konopińska [8, 9].

A shell is a 3D solid body identified in the undeformed placement with a region B of the physical space $\mathcal{E}$ having the translation vector space E. The position vectors $\mathbf{x}$ and $\mathbf{y} = \chi(\mathbf{x})$ relative to some origin $\mathrm{o} \in \mathcal{E}$ of any material particle in the undeformed and deformed placement, respectively, are represented by

$$\mathbf{x} = \boldsymbol{x} + \xi \boldsymbol{t}\,, \quad \mathbf{y} = \boldsymbol{y}(\boldsymbol{x}) + \boldsymbol{\zeta}(\boldsymbol{x}, \xi)\,, \quad \boldsymbol{\zeta}(\boldsymbol{x}, 0) = \mathbf{0}\,. \tag{1.1}$$

Here $\boldsymbol{x}$ and $\boldsymbol{y}$ are position vectors of some shell base surfaces M and $N = \chi(M)$, $\boldsymbol{\zeta}$ is a deviation vector from N, $\boldsymbol{n}$ is the unit vector normal to M and orienting it, $\boldsymbol{t}$ is the unit vector not necessary normal to M with $\boldsymbol{t} \cdot \boldsymbol{n} > 0$, and $\xi \in [-h^-, h^+]$ is the distance from M along $\boldsymbol{t}$ with $h = h^- + h^+$ the initial shell thickness measured along ξ.

The shell base surface M may be irregular one, in general, consisting of regular parts $M_1, M_2, ..., M_n$ joined together along some parts of their edges. The junction curves form together a net of singular surface curves Γ along which the junction jump (or continuity) conditions should be formulated.

The resultant 2D equilibrium equations in the referential description, which are satisfied for any part $\Pi \in M \setminus \Gamma$, are

$$Div_s \boldsymbol{N} + \boldsymbol{f} = \boldsymbol{0}, \quad Div_s \boldsymbol{M} + \text{ax}\left(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T\right) + \boldsymbol{c} = \boldsymbol{0}, \tag{1.2}$$

where $(\boldsymbol{N}, \boldsymbol{M}) \in E \otimes T_x M$ are the referential stress resultant and stress couple tensors, $\boldsymbol{f}$ and $\boldsymbol{c}$ are the external resultant surface force and couple vectors applied on N, but measured per unit area of M, $\boldsymbol{F} = Grad_s \boldsymbol{y} \in E \otimes T_x M$ is the shell deformation gradient tensor, Div_s is the divergence surface operator on M, and $\text{ax}(\cdot)$ is the axial vector associated with the skew tensor $(\cdot)$.

The resultant static boundary conditions satisfied along ∂M_f are

$$\boldsymbol{n}^* - \boldsymbol{N}\boldsymbol{\nu} = \boldsymbol{0}, \quad \boldsymbol{m}^* - \boldsymbol{M}\boldsymbol{\nu} = \boldsymbol{0}, \tag{1.3}$$

where $\boldsymbol{n}^*$ and $\boldsymbol{m}^*$ are the external resultant boundary force and couple vectors applied along $\partial N_f = \chi(\partial M_f)$ but measured per unit length of ∂M_f having the outward unit normal $\boldsymbol{\nu}$, and $(\cdot)^*$ means the prescribed field.

There may be in general k shell elements with regular base surfaces $M_1, M_2, ..., M_n$ joined together by parts of their edges $\partial M_i, i = 1, 2, ..., k \le n,$ along the singular curve Γ. In such a general case the resultant static continuity conditions across the curve Γ are [6, 7]

$$[\boldsymbol{N}\boldsymbol{\nu}] + \boldsymbol{f}_\Gamma = \boldsymbol{0}, \quad [\boldsymbol{M}\boldsymbol{\nu}] + [\boldsymbol{y} \times \boldsymbol{N}\boldsymbol{\nu}] + \boldsymbol{c}_\Gamma = \boldsymbol{0}, \tag{1.4}$$

$$[\boldsymbol{N}\boldsymbol{\nu}] = \sum_{i=1}^{k \le n} \boldsymbol{N}_i \boldsymbol{\nu}_i, \quad [\boldsymbol{M}\boldsymbol{\nu}] = \sum_{i=1}^{k \le n} \boldsymbol{M}_i \boldsymbol{\nu}_i, \tag{1.5}$$

where $[\boldsymbol{N}\boldsymbol{\nu}]$ and $[\boldsymbol{M}\boldsymbol{\nu}]$ are the jumps of $\boldsymbol{N}\boldsymbol{\nu}$ and $\boldsymbol{M}\boldsymbol{\nu}$ at each regular point of Γ, and $\boldsymbol{f}_\Gamma, \boldsymbol{c}_\Gamma$ are some 1D compensating force and couple vector fields applied along $\Gamma \cap \Pi$.

Explicit definitions for $\boldsymbol{f}_\Gamma, c_\Gamma$ in case of branching and self-intersecting shells are given in [7].

2. Kinematic relations at the shell junction

In order to keep the junction relations in focus, we discuss here only two shell elements with regular base surfaces M_1 and M_2 connected together along their common edges coinciding with Γ, see Fig. 1.

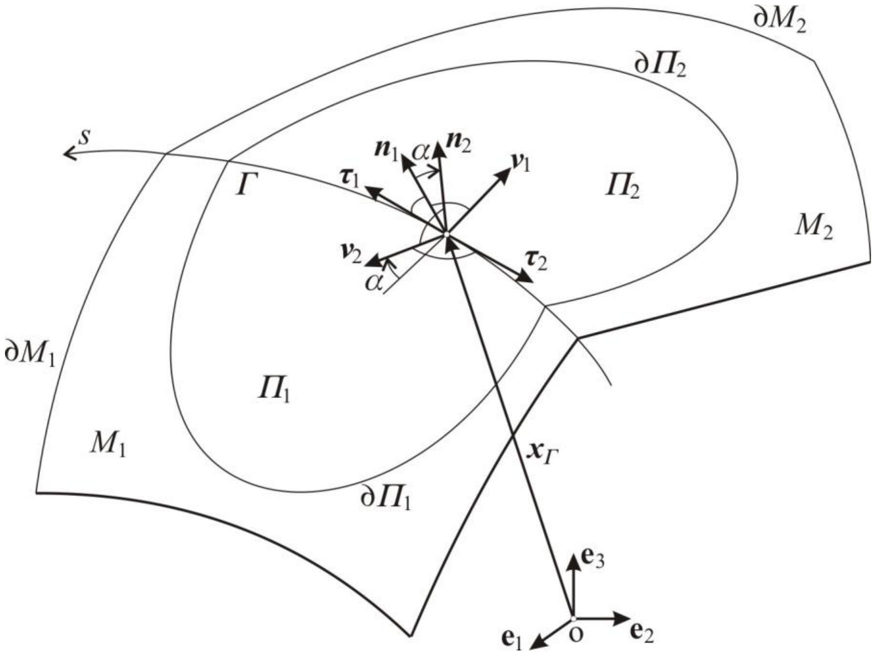

Figure 1. The irregular surface $M = M_1 \cup M_2$ with the fold Γ, [9].

If Γ is oriented consistently with M_1, then at any point $x_\Gamma \in \Gamma$ we have $\boldsymbol{\tau}_\Gamma = \boldsymbol{\tau}_1$, $\boldsymbol{n}_\Gamma = \boldsymbol{n}_1$, and $\boldsymbol{\nu}_\Gamma = \boldsymbol{\nu}_1 = \boldsymbol{\tau}_\Gamma \times \boldsymbol{n}_\Gamma$. At the same point $x_\Gamma \in \Gamma$ the orthonormal triad $\boldsymbol{\nu}_2, \boldsymbol{\tau}_2, \boldsymbol{n}_2$ of the edge $\partial\Pi_2 \cap \Gamma \subset \partial M_2$ does not coincide with the triad $\boldsymbol{\nu}_1, \boldsymbol{\tau}_1, \boldsymbol{n}_1$. In fact, we have,

$$\boldsymbol{\tau}_2 = -\boldsymbol{\tau}_1 , \quad \boldsymbol{n}_2 = \boldsymbol{n}_1 \cos\alpha + \boldsymbol{\nu}_1 \sin\alpha , \quad \boldsymbol{\nu}_2 = -\boldsymbol{\nu}_1 \cos\alpha + \boldsymbol{n}_1 \sin\alpha . \tag{1.6}$$

Hence, in this case the static jumps in (1.5) across Γ are defined as follows:

$$\begin{aligned} [\boldsymbol{N\nu}] &= \boldsymbol{N}_1\boldsymbol{\nu}_1 + \boldsymbol{N}_2\boldsymbol{\nu}_2 = -\left(\boldsymbol{N}_2 \cos\alpha - \boldsymbol{N}_1\right)\boldsymbol{\nu}_1 , \\ [\boldsymbol{M\nu}] &= \boldsymbol{M}_1\boldsymbol{\nu}_1 + \boldsymbol{M}_2\boldsymbol{\nu}_2 = -\left(\boldsymbol{M}_2 \cos\alpha - \boldsymbol{M}_1\right)\boldsymbol{\nu}_1 . \end{aligned} \tag{1.7}$$

For only two surface elements connected along the fold Γ, the direct through-the-thickness integration of 3D equilibrium equations along the skew coordinate ξ indicated in

[7] can be performed *exactly*. It is only sufficient to understand the transverse coordinate ξ as a rectilinear one, not necessarily orthogonal to the surfaces M_1 and M_2, but which coincides on the both sides of the junction region. In the case of only two shell elements there is also no necessity of introducing additional compensating force $\boldsymbol{f}_\Gamma$ and couple $\boldsymbol{c}_\Gamma$ vectors, as well as there is no additional concentrated loadings $\boldsymbol{n}_i, \boldsymbol{m}_i$ and $\boldsymbol{n}_e, \boldsymbol{m}_e$ at the initial and end points of Γ within M as well, which were necessary in the case of branching and self-intersecting shells, see [7]. As a result, in this case the static continuity conditions (1.4) are simplified to

$$[\boldsymbol{N\nu}] = \boldsymbol{0}, \quad [\boldsymbol{M\nu}] + [\boldsymbol{y} \times \boldsymbol{N\nu}] = \boldsymbol{0}, \tag{1.8}$$

where the jumps are defined by (1.7).

In this paper we additionally assume the deformed position vector field $\boldsymbol{y}$ to be always smooth, so that $[\boldsymbol{y}] = \boldsymbol{0}$ across Γ. By this requirement we prevent the shell to be decomposed along Γ during deformation. As a result, the static continuity conditions (1.8) are reduced to

$$[\boldsymbol{N\nu}] = \boldsymbol{0}, \quad [\boldsymbol{M\nu}] = \boldsymbol{0}. \tag{1.9}$$

Let $(\boldsymbol{v}, \boldsymbol{w}) \in E$ be two vector fields smooth at the regular points of $M \setminus \Gamma$, and $(\boldsymbol{v}_\Gamma, \boldsymbol{w}_\Gamma) \in E$ be two other vector fields smooth along Γ. Then for any part $\Pi \subset M$ containing the fold Γ we can set the integral identity

$$\begin{aligned} &\iint_{\Pi \setminus \Gamma} \left\{ \left(Div_s \boldsymbol{N} + \boldsymbol{f} \right) \cdot \boldsymbol{v} + \left(Div_s \boldsymbol{M} + \mathrm{ax}(\boldsymbol{NF}^T - \boldsymbol{FN}^T) + \boldsymbol{c} \right) \cdot \boldsymbol{w} \right\} da \\ &+ \int_{\Pi \cap \partial M_f} \left\{ \left(\boldsymbol{n}^* - \boldsymbol{N\nu} \right) \cdot \boldsymbol{v} + \left(\boldsymbol{m}^* - \boldsymbol{M\nu} \right) \cdot \boldsymbol{w} \right\} ds \\ &- \int_{\Pi \cap \Gamma} \left\{ [\boldsymbol{N\nu}] \cdot \boldsymbol{v}_\Gamma + [\boldsymbol{M\nu}] \cdot \boldsymbol{w}_\Gamma \right\} ds = 0 . \end{aligned} \tag{1.10}$$

By simple algebra we have

$$\begin{aligned} &(Div_s \boldsymbol{N}) \cdot \boldsymbol{v} = \boldsymbol{N} \bullet Grad_s \boldsymbol{v}, \quad (Div_s \boldsymbol{M}) \cdot \boldsymbol{w} = \boldsymbol{M} \bullet Grad_s \boldsymbol{w}, \\ &\mathrm{ax}(\boldsymbol{NF}^T - \boldsymbol{FN}^T) \cdot \boldsymbol{w} = \boldsymbol{N} \bullet (\boldsymbol{WF}), \end{aligned} \tag{1.11}$$

where $\bullet$ is the scalar product in the tensor space such that for any $\boldsymbol{A}, \boldsymbol{B} \in E \otimes T_x M$ we have $\boldsymbol{A} \bullet \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}^T \boldsymbol{B})$, and $\boldsymbol{W} = \boldsymbol{w} \times \boldsymbol{1}$ is the skew tensor, where $\boldsymbol{1}$ means the identity tensor of the space $E \otimes E$.

Since Π is an arbitrarily chosen part of M containing Γ, transforming (1.10) with the help of (1.11) and applying the surface divergence theorems (see [7], f. (23)-(26)) we obtain

$$
\begin{aligned}
&-\iint_{M\setminus\Gamma}\left\{\boldsymbol{N}\bullet\left(Grad_s\boldsymbol{v}-\boldsymbol{WF}\right)+\boldsymbol{M}\bullet Grad_s\boldsymbol{w}\right\}da+\iint_{M\setminus\Gamma}\left(\boldsymbol{f}\cdot\boldsymbol{v}+\boldsymbol{c}\cdot\boldsymbol{w}\right)da\\
&\quad+\int_{\partial M_f}\left(\boldsymbol{n}^*\cdot\boldsymbol{v}+\boldsymbol{m}^*\cdot\boldsymbol{w}\right)ds+\int_{\partial M_d}\left(\boldsymbol{N\nu}\cdot\boldsymbol{v}+\boldsymbol{M\nu}\cdot\boldsymbol{w}\right)ds\\
&\quad+\int_{\Gamma}\left\{[\boldsymbol{N\nu}\cdot\boldsymbol{v}]-[\boldsymbol{N\nu}]\cdot\boldsymbol{v}_\Gamma+[\boldsymbol{M\nu}\cdot\boldsymbol{w}]-[\boldsymbol{M\nu}]\cdot\boldsymbol{w}_\Gamma\right\}ds=0.
\end{aligned}
\tag{1.12}
$$

The real shell deformation is described by the translation vector $\boldsymbol{u}=\boldsymbol{y}-\boldsymbol{x}$ and the rotation tensor $\boldsymbol{Q}=\boldsymbol{d}_i\otimes\boldsymbol{t}_i$, $i=1,2,3,$ of M, where $\boldsymbol{d}_i,\boldsymbol{t}_i$ are orthonormal triads of directors associated with M in the deformed and undeformed placements, respectively. Then the vectors $\boldsymbol{v},\boldsymbol{w}$ in (1.12) may be interpreted as the kinematically admissible virtual translations and rotations,

$$
\boldsymbol{v}\equiv\delta\boldsymbol{u},\quad \boldsymbol{W}=(\delta\boldsymbol{Q})\boldsymbol{Q}^T\equiv\boldsymbol{\Omega}=\boldsymbol{\omega}\times\mathbf{1},\quad \boldsymbol{w}=\frac{1}{2}\left(\mathbf{1}\times\mathbf{1}\right)\bullet\boldsymbol{W}\equiv\boldsymbol{\omega}=\frac{1}{2}\left(\mathbf{1}\times\mathbf{1}\right)\bullet\boldsymbol{\Omega}, \tag{1.13}
$$

where δ is the symbol of virtual change (variation).

Since $\delta\boldsymbol{u}=\boldsymbol{\omega}=\mathbf{0}$ along $\partial M_d=\partial M\setminus\partial M_f$, the integral over ∂M_d in (1.12) vanishes identically. Moreover, it was found in [6, 10] that

$$
Grad_s\delta\boldsymbol{u}-\boldsymbol{\Omega F}=\delta^c\boldsymbol{E},\quad Grad_s\boldsymbol{\omega}=\delta^c\boldsymbol{K}, \tag{1.14}
$$

where $\delta^c(\cdot)=\boldsymbol{Q}\delta\{\boldsymbol{Q}^T(\cdot)\}$ is the co-rotational variation of $(\cdot)$, and the 2D shell stretch and bending tensors are defined by

$$
\boldsymbol{E}=\boldsymbol{J}\mathbf{F}-\boldsymbol{QI},\quad \boldsymbol{K}=\boldsymbol{C}\mathbf{F}-\boldsymbol{QB}. \tag{1.15}
$$

In (1.15), $\boldsymbol{I}=Grad_s\boldsymbol{x}\in E\otimes T_xM$ and $\boldsymbol{J}=grad_s\boldsymbol{y}\in E\otimes T_yN$ are the inclusion operators on $M\setminus\Gamma$ and $N\setminus\chi(\Gamma)$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are the structure curvature tensors of the shell base surface in the undeformed and deformed placement, respectively, and $\mathbf{F}\in T_yN\otimes T_xM$ is the tangential surface deformation gradient such that $dy=\mathbf{F}dx$.

Introducing the virtual strain energy density in $M\setminus\Gamma$ defined by

$$
\sigma=\boldsymbol{N}\bullet\delta^c\boldsymbol{E}+\boldsymbol{M}\bullet\delta^c\boldsymbol{K}, \tag{1.16}
$$

the principle of virtual work following from (1.12) takes the form

$$
\begin{aligned}
\iint_{M\setminus\Gamma}\sigma\,da&=\iint_{M\setminus\Gamma}\left(\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{c}\cdot\boldsymbol{\omega}\right)da+\int_{\partial M_f}\left(\boldsymbol{n}^*\cdot\delta\boldsymbol{u}+\boldsymbol{m}^*\cdot\boldsymbol{\omega}\right)ds\\
&\quad+\int_{\Gamma}\left\{[\boldsymbol{N\nu}\cdot\delta\boldsymbol{u}]-[\boldsymbol{N\nu}]\cdot\boldsymbol{v}_\Gamma+[\boldsymbol{M\nu}\cdot\boldsymbol{\omega}]-[\boldsymbol{M\nu}]\cdot\boldsymbol{w}_\Gamma\right\}ds.
\end{aligned}
\tag{1.17}
$$

The curvilinear integral over Γ in (1.17) includes the jump terms which describe the shell – junction interaction between two joined shell elements with base surfaces M_1 and M_2.

Explicit expressions of the jump terms depend on the type of junction modelled by this approach.

The large variety of types of 1D structural elements, which can be used as junctions in compound shell structures, together with complex kinematics required within the resultant six-field shell model, makes the general non-linear BVP of such structures to be very complex and hardly readable in engineering applications.

The compound jump terms in (1.17) can be decomposed as follows:

$$\begin{aligned}[\boldsymbol{N\nu}\cdot\delta\boldsymbol{u}] &= [\boldsymbol{N\nu}]\cdot<\delta\boldsymbol{u}>+<\boldsymbol{N\nu}>\cdot[\delta\boldsymbol{u}],\\ [\boldsymbol{M\nu}\cdot\boldsymbol{\omega}] &= [\boldsymbol{M\nu}]\cdot<\boldsymbol{\omega}>+<\boldsymbol{M\nu}>\cdot[\boldsymbol{\omega}],\end{aligned} \tag{1.18}$$

where $<\boldsymbol{a}>$ is the average value of $\boldsymbol{a}\in E$ at Γ. In our special case of smooth translations everywhere discussed here, the translation at the junction curve Γ may be interpreted as the average translation of both edges $\partial M_1\cap\Gamma$ and $\partial M_2\cap\Gamma$, so that $<\delta\boldsymbol{u}>\equiv\delta\boldsymbol{u}_\Gamma$. But the rotation tensors $\boldsymbol{Q}_1=\boldsymbol{Q}|_{\partial M_1\cap\Gamma}$ and $\boldsymbol{Q}_2=\boldsymbol{Q}|_{\partial M_2\cap\Gamma}$ of the edges at the same $x_\Gamma\in\Gamma$ may be different, in general, $\boldsymbol{Q}_1\neq\boldsymbol{Q}_2$.

With (1.18) the PVW (1.17) can be reduced to

$$\begin{aligned}\iint_{M\setminus\Gamma}\sigma\,da &= \iint_{M\setminus\Gamma}(\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{c}\cdot\boldsymbol{\omega})da+\int_{\partial M_f}\left(\boldsymbol{n}^*\cdot\delta\boldsymbol{u}+\boldsymbol{m}^*\cdot\boldsymbol{\omega}\right)ds\\ &+\int_\Gamma\left\{[\boldsymbol{M\nu}]\left(<\boldsymbol{\omega}>-\boldsymbol{w}_\Gamma\right)+<\boldsymbol{M\nu}>\cdot[\boldsymbol{\omega}]\right\}ds.\end{aligned} \tag{1.19}$$

Let us introduce explicitly the net rotation tensor $\boldsymbol{Q}_\Gamma$ of Γ such that $\boldsymbol{Q}_2=\boldsymbol{Q}_\Gamma\boldsymbol{Q}_1$ at any $x_\Gamma\in\Gamma$ when x_Γ is approached from both sides of Γ, respectively. Since $\boldsymbol{Q}_2,\boldsymbol{Q}_\Gamma,\boldsymbol{Q}_1$ are all proper orthogonal tensors, we have

$$\boldsymbol{Q}_2\boldsymbol{Q}_2^T=\mathbf{1},\quad \boldsymbol{Q}_\Gamma\boldsymbol{Q}_\Gamma^T=\mathbf{1},\quad \boldsymbol{Q}_1\boldsymbol{Q}_1^T=\mathbf{1}. \tag{1.20}$$

Virtual changes of these orthogonality relations lead to

$$\begin{aligned}\delta\boldsymbol{Q}_2\boldsymbol{Q}_2^T &= -\boldsymbol{Q}_2\delta\boldsymbol{Q}_2^T=\boldsymbol{\omega}_2\times\mathbf{1},\\ \delta\boldsymbol{Q}_\Gamma\boldsymbol{Q}_\Gamma^T &= -\boldsymbol{Q}_\Gamma\delta\boldsymbol{Q}_\Gamma^T=\boldsymbol{\omega}_\Gamma\times\mathbf{1},\\ \delta\boldsymbol{Q}_1\boldsymbol{Q}_1^T &= -\boldsymbol{Q}_1\delta\boldsymbol{Q}_1^T=\boldsymbol{\omega}_1\times\mathbf{1},\end{aligned} \tag{1.21}$$

$$\boldsymbol{\omega}_2=\boldsymbol{\omega}_\Gamma+\boldsymbol{Q}_\Gamma\boldsymbol{\omega}_1. \tag{1.22}$$

The virtual rotations $\boldsymbol{\omega}_2,\boldsymbol{\omega}_\Gamma$ and $\boldsymbol{\omega}_1$ are all defined in the shell deformed placement.

Let the virtual rotation $\boldsymbol{w}_\Gamma$ at Γ be interpreted in terms of $\boldsymbol{\omega}$ as

$$<\boldsymbol{\omega}>=\frac{1}{2}\left\{\boldsymbol{\omega}_\Gamma+(\boldsymbol{Q}_\Gamma+\mathbf{1})\boldsymbol{\omega}_1\right\}\equiv\boldsymbol{w}_\Gamma \tag{1.23}$$

Then the PVW (1.19) can be further reduced to

$$\iint_{M\backslash\Gamma} \sigma\, da = \iint_{M\backslash\Gamma} \left(\boldsymbol{f} \cdot \delta\boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega} \right) da + \int_{\partial M_f} \left(\boldsymbol{n}^* \cdot \delta\boldsymbol{u} + \boldsymbol{m}^* \cdot \boldsymbol{\omega} \right) ds + \int_{\Gamma} < \boldsymbol{M\nu} > \cdot [\boldsymbol{\omega}]\, ds \,. \tag{1.24}$$

The variational statement (1.24) governs the simplified BVP of two regular shell elements with base surfaces M_1 and M_2 joined along the junction Γ. This PVW has been constructed under the assumption that the joint translations are smooth everywhere during deformations. As a result, kinematic description of the junction has been reduced to characterising how the rotations $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ of the neighbouring points of the junction are related to each other during deformation. This still allows one for a variety of possible characterisations of the junction. Some of the simplest particularly appealing junction characterisations are discussed below.

3. Description of the junction

4a. *The stiff junction*

The shell junction along Γ is called *stiff* if the shell deformation is continuous on the whole $M = M_1 \cup M_2$ including Γ. In this case

$$[\delta\boldsymbol{u}] = \boldsymbol{0}\,, \quad [\boldsymbol{\omega}] = \boldsymbol{0}\,, \quad \boldsymbol{u}_1 = \boldsymbol{u}_2\,, \quad \boldsymbol{Q}_1 = \boldsymbol{Q}_2\,, \tag{1.25}$$

and the curvilinear integral along Γ in (1.24) vanishes. The correspondingly simplified PVW is reduced to

$$\iint_{M} \sigma\, da = \iint_{M} \left(\boldsymbol{f} \cdot \delta\boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega} \right) da + \int_{\partial M_f} \left(\boldsymbol{n}^* \cdot \delta\boldsymbol{u} + \boldsymbol{m}^* \cdot \boldsymbol{\omega} \right) ds \,. \tag{1.26}$$

The physical meaning of (1.26) is that in this case the junction along Γ does not contribute to the virtual work of the compound shell structure. The mechanical behaviour of the junction itself is enforced by the behaviour of stiffly joined shell lateral boundary surfaces of regular shell parts with surface elements M_1 and M_2. This is exactly the case of almost all types of shell junctions reviewed in [1]. In particular, within the resultant non-linear six-field shell theory several folded and multi-shell structures with stiff junctions were modelled and analysed with FEM by Chróścielewski et al. [11]. Non-linear dynamic problems of such structures were discussed by Chróścielewski et al. [12]. A number of linear and non-linear FE solutions of multi-shells with stiff junctions was summarized in the book [6].

4b. *The hinge junction*

The *hinge* junction along Γ is understood when $\boldsymbol{u}$ is continuous across Γ, that is $[\delta \boldsymbol{u}] = \mathbf{0}$, $\boldsymbol{u}_1 = \boldsymbol{u}_2$, but $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ are entirely unconstrained when approaching Γ along a path on corresponding M_1, M_2. In this case $[\boldsymbol{\omega}] \neq \mathbf{0}$, in general. However, in order the entirely unconstrained rotations $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ to happen, from equilibrium it follows that no moments at both sides of Γ should be allowed,

$$\boldsymbol{M}_1 \boldsymbol{\nu}_1 = \mathbf{0}, \quad \boldsymbol{M}_2 \boldsymbol{\nu}_2 = \mathbf{0}, \qquad (1.27)$$

so that $< \boldsymbol{M}\boldsymbol{\nu} >= \mathbf{0}$ and hence $< \boldsymbol{M}\boldsymbol{\nu} > \cdot [\boldsymbol{\omega}] = \mathbf{0}$ along Γ. As a result, in the corresponding PVW (1.24) the curvilinear integral along Γ vanishes as well reducing it again formally to (1.26). However, the important difference to the stiff junction is that in the case of hinge junction along Γ the additional static equilibrium conditions (1.27) have to be enforced in the process of solution of BVP.

Some special cases of such BVP for the linear elastic plate problems were proposed in the literature. For example, linear boundary value problems for anisotropic elastic Kirchoff plates with internal line hinges were discussed by Grossi [13]. Natural vibrations of the rectangular plate with a hinge line were analysed by Huang et al. [14] and Grossi and Raffo [15] within the linear Kirchhoff plate model, while Xiang and Reddy [16] used the linear first order shear deformation plate model for this purpose.

4c. *The deformable junction*

In the PVW (1.24) both ingredients $< \boldsymbol{M}\boldsymbol{\nu} >$ and $[\boldsymbol{\omega}]$ in the last integral may not together identically vanish, in general, that is $< \boldsymbol{M}\boldsymbol{\nu} > \neq \mathbf{0}$ and $[\boldsymbol{\omega}] \neq \mathbf{0}$. In this general case the shell junction along Γ may be called *deformable.*

From engineering point of view, the junctions can be classified according to:

- the type of medium used: bolted, welded, riveted, glued, adhesively bonded etc.;
- the type of internal forces the junction is expected to transmit: membrane, shear, moment (stiff, deformable);
- the type of elements the junction is joining: regular shell elements, transition stiffening ringbeams, special junction constructions.

This leads to a large variety of constructions of junctions in compound shell structures. Mechanical and/or deformability properties of each particular case of such junction should be known in advance before the analyses take place.

Let us differentiate the orthogonality relations (1.21) along Γ ,

$$\begin{aligned} &(\boldsymbol{Q}_2)'\boldsymbol{Q}_2^T = -\boldsymbol{Q}_2(\boldsymbol{Q}_2^T)' = \boldsymbol{\kappa}_2 \times \mathbf{1}, \\ &(\boldsymbol{Q}_\Gamma)'\boldsymbol{Q}_\Gamma^T = -\boldsymbol{Q}_\Gamma(\boldsymbol{Q}_\Gamma^T)' = \boldsymbol{\kappa}_\Gamma \times \mathbf{1}, \\ &(\boldsymbol{Q}_1)'\boldsymbol{Q}_1^T = -\boldsymbol{Q}_1(\boldsymbol{Q}_1^T)' = \boldsymbol{\kappa}_1 \times \mathbf{1}, \quad (.)' = \frac{d}{ds}(.)\,, \end{aligned} \tag{1.28}$$

$$\boldsymbol{\kappa}_2 = \boldsymbol{\kappa}_\Gamma + \boldsymbol{Q}_\Gamma \boldsymbol{\kappa}_1 \,. \tag{1.29}$$

The vector $\boldsymbol{\kappa}_\Gamma$ describes the bending properties of the junction curve Γ during shell deformation.

The mechanical behaviour of the deformable junction can be characterized by the relation

$$< \boldsymbol{M\nu} > = f(\boldsymbol{\kappa}_\Gamma)\,, \tag{1.30}$$

where f is a smooth vector function of vectorial argument at any $x_\Gamma \in \Gamma$. The relation (1.30) is the kind of 1D constitutive equation modelling deformability properties of real engineering junctions. It is apparent that due to possible complexity of engineering junction constructions the function f should be established from appropriate experiments for each particular type of the junction.

With (1.30) and (1.22) the PVW (1.19) takes the modified form

$$\begin{aligned} \iint_{M\setminus\Gamma} \sigma\, da = \iint_{M\setminus\Gamma} \left(\boldsymbol{f} \cdot \delta\boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega} \right) da + \int_{\partial M_f} \left(\boldsymbol{n}^* \cdot \delta\boldsymbol{u} + \boldsymbol{m}^* \cdot \boldsymbol{\omega} \right) ds \\ + \int_\Gamma f(\boldsymbol{\kappa}_\Gamma) \cdot \left\{ \boldsymbol{\omega}_\Gamma + (\boldsymbol{Q}_\Gamma + \mathbf{1})\boldsymbol{\omega}_1 \right\} ds\,. \end{aligned} \tag{1.31}$$

If there exists a scalar function $W(\boldsymbol{\kappa}_\Gamma)$ such that $f(\boldsymbol{\kappa}_\Gamma) = \partial W / \partial \boldsymbol{\kappa}_\Gamma$, the junction along Γ may be called *elastic*. The function W may be quite complex non-linear function of $\boldsymbol{\kappa}_\Gamma$, so that such a junction is *non-linearly elastic*, in general. But in some cases W may become a quadratic function such that

$$W = \frac{1}{2} \boldsymbol{L} \bullet (\boldsymbol{\kappa}_\Gamma \otimes \boldsymbol{\kappa}_\Gamma)\,, \quad f(\boldsymbol{\kappa}_\Gamma) = \boldsymbol{L}\boldsymbol{\kappa}_\Gamma\,, \tag{1.32}$$

where $\boldsymbol{L}$ is the 2^{nd} order tensor of rotational material properties along Γ . In this case the shell junction can be called *linearly elastic.*

Special cases of elastic junction conditions within the linear Kirchhoff-type theory of elastic plates were discussed by Bernadou [17], Titaux and Sanchez-Palencia [18] and Nardinocci [19]. Elastic junctions between two thin linearly elastic shells of Koiter type were asymptotically analysed by Akian [20] and Merabet et al. [21]. Within the non-linear theory of thin shells of Kirchhoff-Love type, description of several types of shell junctions were given by Makowski et al. [22], and explicit numerical solutions of the shell of revolution with deformable elasto-plastic junctions were given by Chróścielewski et al. [23, 24]. Within the non-linear six-field shell theory, the deformable junction of one branch at the shell branching was kinematically classified in [8] as locally elastic, non-locally elastic and dissipative.

4. Conclusions

Within the general non-linear resultant shell theory, we have formulated the virtual work principle for two regular shell elements joined together along their common boundaries. It has been assumed that translations across the junction curve are smooth, but no restrictions are enforced for the rotations. It has been shown that for stiff and hinge type junctions the curvilinear integral along the junction curve vanishes identically and does not bring an additional virtual work to the shell BVP. In the case of deformable junction, the 1D constitutive type relation has been proposed for the junction moments in terms of net rotations of the junction curve. The constitutive function should be established by experiments for each engineering construction of the deformable junction. As special cases, description of non-linearly elastic and linearly elastic junctions have been noted.

The proposed description of shell junctions should allow development of appropriate numerical FEM programs for non-linear analyses of multi-shells with various types of junctions.

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