On the resultant six-field linear theory of elastic shells

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ABSTRACT: As compared with the classical linear shell models of Kirchhoff-Love and Timoshenko-Reissner type, the six-field linear shell model contains the drilling rotation as an independent kinematic variable as well as two surface drilling couples with two work-conjugate surface drilling bending measures are present in description of the shell stress-strain state. Within the six-field linear theory of elastic shells the following new results are presented: 1) the extended static-geometric analogy; 2) the complex BVP for complex independent variables; 3) description of deformation of the shell boundary element; 4) the Cesáro type formulas and expressions for the vectors of stress functions along the shell boundary contour; 5) estimates of surface gradients of 2D stress and/or strain measures to be of 2nd order small in the 2nd approximation to the complementary energy density of the elastic shell. These theoretical results should be of interest to specialists of the linear theory of shells and those developing computer FEM software for analyses of irregular multi-shell structures.

1 INTRODUCTION

The resultant non-linear theory of elastic shells was proposed by Reissner (1974), developed in a number of papers and summarised in several monographs, for example by Libai and Simmonds (1998) and Chróścielewski et al. (2004). In this formulation the 2D non-linear shell equilibrium equations are derived by the *exact* through-the-thickness integration of equilibrium conditions of non-linear elasticity. Then the 2D virtual work identity allows one to construct *uniquely* the 2D shell kinematics consisting of the translation vector $\boldsymbol{u}$ and rotation tensor $\boldsymbol{Q}$ (or an equivalent finite rotation vector $\boldsymbol{\psi}$) fields (six independent scalar variables) defined on the shell base surface. The 2D surface stretch and bending measures follow then again *uniquely* as direct consequence of the exact resultant equilibrium equations. When such a resultant shell model is linearized for infinitesimal translations, rotations, stretches and bending measures, the linearized drilling rotation remains as the independent kinematic variable, as well as two linearized drilling couples and two work-conjugate drilling bending measures remain in the description of 2D stress and strain state. The latter features contradict all classical shell formulations of the Kirchhoff-Love and Timoshenko-Reissner type following from linear elasticity by any 3D-to-2D reduction technique.

In this note we wish to investigate what the resultant six-field linear theory of shells brings to the classical linear shell models.

2 EXTENDED STATIC–GEOMETRIC ANALOGY

For the general system of notation let me refer to the recent paper by Pietraszkiewicz & Konopińska (2014), where the exact resultant 2D relations of the six-field non-linear theory of shells are briefly recalled. When translations and rotations of the shell base surface M are assumed to be small, the corresponding vectors $\boldsymbol{u},\boldsymbol{\psi}$ on M are given by

$$\boldsymbol{u} = u_\alpha \boldsymbol{a}^\alpha + w\boldsymbol{n}\,, \quad \boldsymbol{\psi} = \boldsymbol{n}\times(\psi_\alpha \boldsymbol{a}^\alpha) + \psi\boldsymbol{n}$$

where $\boldsymbol{a}^\alpha, \boldsymbol{n}$ are the contravariant base vectors of M. The above relations contain the drilling rotation ψ which is not present in the classical shell models.

The corresponding vectorial surface strain measures are

$$\boldsymbol{\varepsilon}_\alpha = \boldsymbol{u},_\alpha - \boldsymbol{\psi}\times\boldsymbol{a}_\alpha = E_{\alpha\beta}\boldsymbol{a}^\beta + E_\alpha \boldsymbol{n}$$

$$\boldsymbol{\kappa}_\alpha = \boldsymbol{\psi},_\alpha = \varepsilon_{\lambda\beta} K_\alpha^{\cdot\lambda} \boldsymbol{a}^\beta + K_\alpha \boldsymbol{n}$$

The resultant vectorial surface stress measures are

$$\boldsymbol{n}^\alpha = N^{\alpha\beta}\boldsymbol{a}_\beta + Q^\alpha \boldsymbol{n}\,, \quad \boldsymbol{m}^\alpha = \varepsilon_{\lambda\beta} M^{\alpha\lambda}\boldsymbol{a}^\beta + M^\alpha \boldsymbol{n}$$

Linearization of the component form of exact equilibrium equations gives

$$N^{\alpha\beta}|_{\alpha} - b^{\beta}_{\alpha} Q^{\alpha} + f^{\beta} = 0\,, \quad Q^{\alpha}|_{\alpha} + b_{\alpha\beta} N^{\alpha\beta} + f = 0$$
$$M^{\alpha\beta}|_{\alpha} - Q^{\beta} + \varepsilon^{\lambda\beta} b_{\alpha\lambda} M^{\alpha} + m^{\beta} = 0 \qquad (1)$$
$$M^{\alpha}|_{\alpha} + \varepsilon_{\alpha\beta}(N^{\alpha\beta} - b^{\alpha}_{\lambda} M^{\lambda\beta}) + m = 0$$

These twelve PDEs involve the drilling stress couples M^{α} as well, which are not present in analogous PDEs of the classical linear shell models of the K-L and T-R type.

The corresponding exact 2D compatibility conditions of the six-field non-linear theory of shells follow from integrability conditions $\varepsilon^{\alpha\beta}\boldsymbol{u}_{|\alpha\beta} = \mathbf{0}$ for the surface translation vector and $\varepsilon^{\alpha\beta}\boldsymbol{Q}_{|\alpha\beta} = \mathbf{0}$ for the surface rotation tensor fields. For small stretch and bending surface measures the component form of non-linear compatibility conditions can be linearized to

$$\varepsilon^{\alpha\beta}\left(E_{\alpha\lambda|\beta} - E_{\alpha} b_{\beta\lambda} + \varepsilon_{\alpha\lambda} K_{\beta}\right) = 0$$
$$\varepsilon^{\alpha\beta}\left(E_{\alpha|\beta} + E_{\alpha\lambda} b^{\lambda}_{\beta} + K_{\alpha\beta}\right) = 0 \qquad (2)$$
$$\varepsilon^{\alpha\beta}\left(\varepsilon^{\rho\lambda} K_{\alpha\rho|\beta} + b^{\lambda}_{\alpha} K_{\beta}\right) = 0, \varepsilon^{\alpha\beta}\left(K_{\alpha|\beta} + \varepsilon^{\lambda\rho} K_{\alpha\lambda} b_{\beta\rho}\right) = 0$$

These twelve PDEs involve the 2D surface drilling bending measures K_{α}. Again, the fields K_{α} are not present in compatibility conditions of any classical linear shell models.

Between the homogeneous equilibrium equations (1) and the compatibility conditions (2) there exists the following correspondence:

$$N^{\alpha\beta} \Leftrightarrow \varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda} K_{\rho\lambda}\,, \qquad Q^{\alpha} \Leftrightarrow -\varepsilon^{\alpha\rho} K_{\rho}$$
$$M^{\alpha\beta} \Leftrightarrow -\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda} E_{\rho\lambda}\,, \quad M^{\alpha} \Leftrightarrow -\varepsilon^{\alpha\rho} E_{\rho} \qquad (3)$$

When the resultant stress measures in (1) are replaced by the 2D strain measures according to (3), the homogeneous equilibrium equations (1) are converted exactly to the compatibility conditions (2). The correspondence (3) can be called the *extended static-geometric analogy* in the resultant six-field linear theory of shells.

The property (3) allows one to introduce the 2D complex stress and strain measures in analogy to those used by Novozhilov (1964) in the classical K-L type linear shell theory,

$$\tilde{N}^{\alpha\beta} = N^{\alpha\beta}_{*} - i\,Ehc\,\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\tilde{K}_{\rho\lambda}$$
$$\tilde{Q}^{\alpha} = Q^{\alpha}_{*} + i\,Ehc\,\varepsilon^{\alpha\rho}\tilde{K}_{\rho}$$
$$\tilde{M}^{\alpha\beta} = M^{\alpha\beta}_{*} + i\,Ehc\,\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\tilde{E}_{\rho\lambda}$$
$$\tilde{M}^{\alpha} = M^{\alpha}_{*} + i\,Ehc\,\varepsilon^{\alpha\rho}\tilde{E}_{\rho}$$

where $i = \sqrt{-1}$, $c = h/\sqrt{12(1-\nu^2)}$, the star means a particular solution of inhomogeneous PDEs (1), and $\tilde{K}_{\rho\lambda}, \tilde{K}_{\rho}, \tilde{E}_{\rho\lambda}, \tilde{E}_{\rho}$ are the 2D strain measures but constructed from the complex translations and rotations,

$$\tilde{u}_{\alpha} = u_{\alpha} + i\,\overline{u}_{\alpha}\,, \;\; \tilde{w} = w + i\,\overline{w}$$
$$\tilde{\psi}_{\alpha} = \psi_{\alpha} + i\,\overline{\psi}_{\alpha}\,, \;\; \tilde{\psi} = \psi + i\,\overline{\psi}$$

where by overbar we denote the stress functions. This leads to the following complex linear equilibrium equations:

$$\tilde{N}^{\alpha\beta}|_{\alpha} - b^{\beta}_{\alpha}\tilde{Q}^{\alpha} + f^{\beta} = 0\,, \; \tilde{Q}^{\alpha}|_{\alpha} + b_{\alpha\beta}\tilde{N}^{\alpha\beta} + f = 0$$
$$\tilde{M}^{\alpha\beta}|_{\alpha} - \tilde{Q}^{\beta} + \varepsilon^{\lambda\beta} b_{\alpha\lambda}\tilde{M}^{\alpha} + m^{\beta} = 0 \qquad (4)$$
$$\tilde{M}^{\alpha}|_{\alpha} + \varepsilon_{\alpha\beta}(\tilde{N}^{\alpha\beta} - b^{\alpha}_{\lambda}\tilde{M}^{\lambda\beta}) + m = 0$$

The system (4) of PDEs is of 6th order in complex domain as compared with PDEs (1) which are of 12th order in real domain. One can apply the complex shell equations (4) with hope to obtain more accurate analytical results than those based on K-L shell model presented in Novozhilov (1964) and Novozhilov et al. (1991) and those based on T-R type shell model presented in Pelekh (1978).

3 RELATIONS ON THE SHELL BOUNDARY

In the resultant shell theory, the undeformed rectilinear shell lateral boundary surface $\partial\mathrm{B}^*$ deforms into the deformed lateral boundary surface $\chi(\partial\mathrm{B}^*)$, which is not the rectilinear one anymore. To describe the linearized deformation of $\partial\mathrm{B}^*$ into $\chi(\partial\mathrm{B}^*)$, let the boundary contour ∂M of the shell base surface be parametrized by the length coordinate s. Along ∂M we can introduce the triad of orthonormal vectors: the tangent $\boldsymbol{\tau}$, the $\boldsymbol{n}$ normal to M, and the exterior normal $\boldsymbol{\nu} = \boldsymbol{\tau}\times\boldsymbol{n}$. In the base $\boldsymbol{\nu}, \boldsymbol{\tau}, \boldsymbol{n}$ the translation $\boldsymbol{u}$ and rotation $\boldsymbol{\psi}$ vectors are represented by

$$\boldsymbol{u} = u_{\nu}\boldsymbol{\nu} + u_{\tau}\boldsymbol{\tau} + w\boldsymbol{n}\,, \quad \boldsymbol{\psi} = -\psi_{\tau}\boldsymbol{\nu} + \psi_{\nu}\boldsymbol{\tau} + \psi\boldsymbol{n} \qquad (5)$$

According to the modified polar decomposition the material fiber $\boldsymbol{\tau}ds$ tangent to ∂M should be rotated by a total rotation vector $\boldsymbol{\psi}_{\tau}$ into the fiber $\hat{\boldsymbol{\tau}}ds$ tangent to $\chi(\partial M)$, where $\hat{\boldsymbol{\tau}}$ denotes the unit vector tangent to the deformed boundary contour. This can be achieved by two rotations: the global rotation vector $\boldsymbol{\psi}$ of the neighborhood of $x \in \partial M$ followed by an additional rotation $\boldsymbol{\psi}_{a}$ associated with the stretch in the direction $\boldsymbol{\tau}$. After transformations given in detail by Pietraszkiewicz (2015) the linearized total rotation vector is given by

$$\boldsymbol{\psi}_{\tau} = \boldsymbol{\psi} + \boldsymbol{\psi}_{a} = (E_{\tau} - \psi_{\tau})\boldsymbol{\nu} + \psi_{\nu}\boldsymbol{\tau} - (E_{\tau\nu} - \psi)\boldsymbol{n} \qquad (6)$$

Differentiation of the translation and rotation fields along ∂M gives

$$\frac{d\boldsymbol{u}}{ds} = E_{\tau\tau}\boldsymbol{\tau} + \boldsymbol{\psi}_{\tau}\times\boldsymbol{\tau}, \quad \boldsymbol{k}_{\tau} = \frac{d\boldsymbol{\psi}_{\tau}}{ds} = -k_{\tau\tau}\boldsymbol{\nu} + k_{\tau\nu}\boldsymbol{\tau} + k_{\tau}\boldsymbol{n}$$

where the components $k_{\tau\tau}, k_{\tau\nu}, k_{\tau}$ are expressed only through the surface strain measures and geometry of ∂M. These relations can be integrated along ∂M to obtain

$$\boldsymbol{u} = \boldsymbol{u}^0 + \int_{s_0}^{s} \left(E_{\tau\tau}\boldsymbol{\tau}' + \boldsymbol{\psi}_\tau \times \boldsymbol{\tau}' \right) ds', \ \boldsymbol{\psi}_\tau = \boldsymbol{\psi}_\tau^0 + \int_{s_0}^{s} \boldsymbol{k}_\tau ds' \quad (7)$$

Using some transformations the relation $(7)_1$ can also be given in the form

$$\boldsymbol{u} = \boldsymbol{u}^0 + \boldsymbol{\psi}_\tau^0 \times (\boldsymbol{x} - \boldsymbol{x}_0) + \int_{s_0}^{s} \left[\left(\int_{s_0}^{s'} \boldsymbol{k}_\tau ds'' \right) \times \boldsymbol{\tau}' + E_{\tau\tau}\boldsymbol{\tau}' \right] ds'$$

The relation above and $(7)_2$, derived here along the boundary contour ∂M, are also valid for any regular curve C on the shell base surface. Since for the simply connected base surface M the resulting translations and rotations do not depend on the type of surface curve connecting the initial and current points on M, these relations can be treated as the Cesáro type formulas in the resultant linear six-field theory of shells.

When solving shell problems in terms of stress functions it is of importance to know how the stress function vectors $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{\psi}}$ are related to the resulting force and moment vectors of loads acting along the boundary contour ∂M_f.

The boundary loads $\boldsymbol{n}_\nu = \boldsymbol{n}^\alpha \nu_\alpha$ and $\boldsymbol{m}_\nu = \boldsymbol{m}^\alpha \nu_\alpha$ per unit length of ∂M_f are

$$\boldsymbol{n}_\nu = \boldsymbol{n}_\nu^* - Ehc\,\bar{\boldsymbol{\kappa}}_\tau, \ \boldsymbol{m}_\nu = \boldsymbol{m}_\nu^* - Ehc\left(\bar{E}_{\tau\tau}\boldsymbol{\tau} + \bar{\boldsymbol{\psi}}_a \times \boldsymbol{\tau} \right)$$

where the first terms $\boldsymbol{n}_\nu^*, \boldsymbol{m}_\nu^*$ correspond to the chosen particular solution of the inhomogeneous equilibrium equations (1).

The resulting force vector $\mathbf{F}$ of the loads $\boldsymbol{n}_\nu$ acting along the part $s_0 s$ of ∂M_f is

$$\mathbf{F} = \int_{s_0}^{s} \boldsymbol{n}_\nu ds' = \int_{s_0}^{s} \left(\boldsymbol{n}_\nu^* - Ehc\,\bar{\boldsymbol{\kappa}}_\tau \right) ds'$$

The resulting moment vector $\mathbf{B}$ of the loads $\boldsymbol{n}_\nu$ and $\boldsymbol{m}_\nu$ acting along the part $s_0 s$ of ∂M_f, taken with regard to the current point $x \in \partial M_f$ with the coordinate s, is (see Pietraszkiewicz 2015)

$$\mathbf{B} = \mathbf{B}^* - Ehc\left[\bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}^0 + (\boldsymbol{x} - \boldsymbol{x}_0) \times \bar{\boldsymbol{\psi}}^0 \right]$$

Solving the two relations given above, for the stress function vectors $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{\psi}}$ we obtain

$$\bar{\boldsymbol{\psi}} = \bar{\boldsymbol{\psi}}^0 - \frac{\mathbf{F} - \mathbf{F}^*}{Ehc}, \quad \bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}^0 + \bar{\boldsymbol{\psi}}^0 \times (\boldsymbol{x} - \boldsymbol{x}_0) - \frac{\mathbf{B} - \mathbf{B}^*}{Ehc} \quad (8)$$

The relations (8) along ∂M_f are also valid for any regular curve C on the shell base surface M. In the case of simply connected M and the closed curve C we can assume $\bar{\boldsymbol{u}}^0 = \bar{\boldsymbol{\psi}}^0 = \mathbf{0}$. This means the change of stress function vectors by the term $\bar{\boldsymbol{u}}^0 + \bar{\boldsymbol{\psi}}^0 \times (\boldsymbol{x} - \boldsymbol{x}_0)$ of "rigid-body motion" type which does not influence the stress-deformation state of the shell. In the case of shell problems with non-vanishing differences $\mathbf{F} - \mathbf{F}^*$ and $\mathbf{B} - \mathbf{B}^*$ as well as with a multi-connected M the relations (8) should allow to construct multivalued parts of the stress functions in analogy to procedures developed by Chernykh (1968) and Pietraszkiewicz (1968) for the linear K-L type elastic shells.

4 GRADIENTS OF 2D SHELL MEASURES IN THE RESULTANT STRESS WORKING

From the resultant thermomechanics of shells worked out by Pietraszkiewicz (2011) it follows that even if thermal fields are neglected and kinematic fields are linearized, the constitutive equations of elastic shells are still allowed to depend upon surface gradients of 2D strain measures. We are not aware of any discussion of this problem in the literature.

To have some insight into the problem, let us remind that distribution of translations through the shell thickness is non-linear, in general. For an arbitrary deformation of the shell space, we have introduced in Chróścielewski et al. (2004) the intrinsic deviation vector $\mathbf{e}(\theta^\alpha, \xi)$ defined by

$$\mathbf{e} = \boldsymbol{Q}^T \boldsymbol{\zeta} - \xi \boldsymbol{n} = e^\rho(\xi)\mathbf{g}_\rho(\xi) + e(\xi)\boldsymbol{n} \quad (9)$$

where $\boldsymbol{Q}\mathbf{e}$ is a measure of deviation of the deformed curved material fiber from its approximately linear rotated shape $\xi \boldsymbol{Q}\boldsymbol{n}$, see Figure 4 in Pietraszkiewicz (2015).

The 3D stress power density is defined by $\Sigma = (\mathbf{FS}) : \dot{\mathbf{F}}$, where $\mathbf{F}$ is the 3D deformation gradient, the overdot means the material time derivative, $\mathbf{S} = \mathbf{s}^i \otimes \mathbf{g}_i$ is the 2nd Piola-Kirchhoff stress tensor, and : denotes the double-dot (scalar) product in the tensor space. Using the modified polar decomposition $\mathbf{F} = \boldsymbol{Q}\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda} \neq \boldsymbol{\Lambda}^T$ is the modified stretch tensor, we can calculate exactly the resultant 2D stress power density in the form

$$\mathit{\Sigma} = \int_{-}^{+} \Sigma \mu d\xi = N^{\alpha\beta}\dot{E}_{\alpha\beta} + Q^\alpha \dot{E}_\alpha + M^{\alpha\beta}\dot{K}_{\alpha\beta} + M^\alpha \dot{K}_\alpha$$
$$+ \int_{-}^{+} (\dot{\mathbf{e}} \times \boldsymbol{\Lambda}\mathbf{s}^\alpha) \mu d\xi \bullet \boldsymbol{\kappa}_\alpha + \int_{-}^{+} (\boldsymbol{\Lambda}\mathbf{S}) : \nabla \dot{\mathbf{e}} \mu d\xi \quad (10)$$

where ∇ means the 3D gradient performed in the undeformed shell space. The last two integrals in (10) represent that part of $\mathit{\Sigma}$ which is not expressible through the 2D shell stress and strain measures alone.

In case of linear equilibrium problems, material time derivatives in (10) become linear increments of the fields from the undeformed state and the 2D stress power (10) reduces to the resultant 2D stress working, i.e. to the relation (10) without overdots.

We have thoroughly analyzed the principal terms of the last two integrals in (10) within the accuracy of equilibrium problems of the linear theory of isotropic elastic shells. In particular, we have used some results obtained Rychter (1988) for kinematically admissible components of 3D displacement field of the Reissner type linear shell theory and postulate approximate through-the-thickness distributions of the intrinsic deviation vector

$$e_\rho \simeq k(\xi) q_\rho + g(\xi) c_\rho \,, \quad e \simeq r(\xi) q + s(\xi) d$$

$$q = -\frac{1}{40} h^2 D^{\alpha\beta} K_{(\alpha\beta)} \,, \quad d = -\frac{h^3}{48} D^{\lambda\mu} D^{\alpha\beta} E_{(\alpha\beta)|\lambda\mu}$$

$$r(\xi) = \frac{20\xi^2}{h^2} - 1 \,, \quad s(\xi) = \frac{2}{h}\left(\frac{4\xi^3}{h^2} - \xi\right)$$

while $k(\xi), g(\xi), q_\rho, c_\rho$, and $D^{\alpha\beta}$ are defined in Pietraszkiewicz & Konopińska (2014). Then for the principal terms of $(10)_2$ after appropriate transformations we are able to obtain the following consistently estimated relations:

$$\int_-^+ (\mathbf{e} \times \mathbf{\Lambda s}^\alpha) \mu \mathrm{d}\xi \bullet \boldsymbol{\kappa}_\alpha \simeq \frac{2}{3} q N^{\alpha\beta} K_{(\alpha\beta)} \sim Eh\eta^2 \cdot \left(\frac{1}{60} \nu\eta\right)$$

$$\int_-^+ (\mathbf{\Lambda s}^\alpha) \bullet \mathbf{e},_\alpha \mu \mathrm{d}\xi \simeq \frac{3}{4} Q^\alpha q,_\alpha \sim Eh\eta^2 \cdot \left(\frac{3}{160} \nu\theta \frac{h}{\lambda}\right)$$

$$\int_-^+ (\mathbf{\Lambda s}^3) \bullet \mathbf{e},_3 \mu \mathrm{d}\xi \simeq -9 M^{\alpha\beta}{}_{|\beta\alpha} q - \frac{25}{3} b_{\alpha\beta} N^{\alpha\beta} q \qquad (11)$$

$$\sim Eh\eta^2 \cdot \left(\nu\theta^2, \nu\frac{h^2}{\lambda^2}\right)$$

where θ is a small parameter defined in Pietraszkiewicz & Konopińska (2014), the large parameter $\lambda = \min_{x \in M}\left(L, b, \sqrt{hR}, h/\sqrt{\eta}\right)$, $(\cdot),_\alpha \sim (\cdot)/\lambda$, and $h/\lambda \sim \theta$.

Within the first-approximation linear theory of elastic shells, all terms in (11) can be neglected as compared with the main terms $\sim Eh\eta^2$ in (10). However, in the Timoshenko-Reissner type and the resultant six-field linear shell models the terms (11) are of the same order (aside from small numerical factors) as those following from Q^α. Their role should still be discussed if these linear models are to be regarded as energetically consistent ones.

5 CONCLUSIONS

Within the resultant six-field linear theory of elastic shells we have formulated several results which are not available elsewhere. Among the new results let us point out the following:

- Formulation of the extended static-geometric analogy and derivation of complex shell relations for the complex displacements (section 4).
- Description of infinitesimal deformation of the shell boundary element (section 5) and derivation of corresponding Cesáro type formulas.
- Expressions (8) for the vectors of stress functions in terms of the resulting force and moment vectors acting along the boundary contour and along an arbitrary curve on the shell base surface.
- Estimation that surface gradients of 2D stress and/or strain measures may be of 2nd order small in the 2nd approximation to the complementary energy density.

These theoretical results should be of interest to specialists of the linear theory of shells and to those developing computer FEM software for analyses of irregular multi-shell structures.

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