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Elastic Shells, Material Symmetry Group

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Synonyms

Constitutive equations of elastic shells; Isotropy group of hyperelastic shells

Definition

Constitutive equations of elastic shells follow by differentiation of the two-dimensional strain energy density as a function of shell strain measures. The material symmetry group of the elastic shell consists of an ordered triple of 2nd-order transformation tensors which make the energy function invariant under changes of reference placement of the shell.

Introduction

The general resultant non-linear theory of elastic shells discussed here was proposed by Reissner (1974), developed in a number of papers and summarised in monographs by Libai and Simmonds (1998), Chróścielewski et al. (2004) and Eremeyev and Zubov (2008). Basic relations of this version of shell theory are concisely presented by Pietraszkiewicz (2018) in this Encyclopaedia.

The elastic material behaviour of the shell is characterised by the two-dimensional (2D) strain energy density being a function of 2D shell strain measures. Explicit form of the function depends upon the reference placement of the shell base surface. Analysing invariance properties of the strain energy function under various changes of the reference placement, Eremeyev and Pietraszkiewicz (2006) established and discussed the material symmetry group for this general resultant shell theory. It was found that the group consists of an ordered triple of 2nd-order transformation tensors which make the strain energy function invariant under changes of the reference placement. In particular, it became possible to concisely define the fluid, solid or liquid crystal shells directly in terms of members of the group. In case of solid shells, isotropic, hemitropic, transversally isotropic, orthotropic and other shell material properties were precisely defined. It was also found that some correspondingly reduced representations of the 2D strain energy function, which satisfy appropriate invariance requirements, became of much simpler form and contained much less constitutive constants to be established by experiments.

In this entry, the material symmetry group for the resultant non-linear elastic shell model is briefly presented following the work by Eremeyev and Pietraszkiewicz (2006). However, due to the limited space only basic ideas and definitions as well as some most important relations and results are included here. For more details the interested reader should consult the original work and references given there.

Basic Shell Relations

A shell is a three-dimensional (3D) solid body identified in a reference (undeformed) placement with a region B of the physical space $\mathcal{E}$ with V as its translation vector space. In a given inertial frame $(\mathrm{O},\mathbf{e}_i)$, $i=1,2,3$, where $\mathrm{O}\in\mathcal{E}$ and $\mathbf{e}_i\in V$ are orthonormal vectors, the position vectors $\mathbf{x}$ and $\mathbf{y}=\chi(\mathbf{x})$ of any material particle in the reference and deformed placements, respectively, can conveniently be represented by

$$\mathbf{x}=\boldsymbol{x}+\xi\boldsymbol{n}, \quad \mathbf{y}=\boldsymbol{y}(\boldsymbol{x},t)+\boldsymbol{\zeta}(\boldsymbol{x},\xi), \quad \boldsymbol{\zeta}(\boldsymbol{x},0)=\mathbf{0}. \tag{1}$$

Here $\boldsymbol{x}$ and $\boldsymbol{y}$ are the position vectors of some shell base surfaces M and $N=\chi(M)$ in the reference and deformed placements, respectively, ξ is the distance from M along the unit normal vector $\boldsymbol{n}$ orienting M such that $\xi\in[-h^-,h^+]$, $h=h^-+h^+$ is the undeformed shell thickness, $\boldsymbol{\zeta}$ is a deviation vector of $\mathbf{y}$ from N, while χ and χ mean the 3D and 2D deformation functions, respectively.

Let the vector field $\boldsymbol{u}(\boldsymbol{x})$ represent work-averaged translations of M and the proper orthogonal tensor field $\boldsymbol{Q}(\boldsymbol{x})$ work-averaged rotations of the shell cross sections. The deformed shell configuration can then be described by the relations, see Fig. 1,

$$\boldsymbol{y}=\boldsymbol{x}+\boldsymbol{u}, \quad \boldsymbol{d}_\alpha=\boldsymbol{Q}\boldsymbol{t}_\alpha, \quad \boldsymbol{d}=\boldsymbol{Q}\boldsymbol{n}, \tag{2}$$

where $\boldsymbol{t}_\alpha=\partial\boldsymbol{x}/\partial s_\alpha\equiv\boldsymbol{x},_\alpha$, s_α, $\alpha=1,2$, are surface length coordinates coinciding with lines of principal curvatures on M, while $\boldsymbol{t}_\alpha,\boldsymbol{n}$ and $\boldsymbol{d}_\alpha,\boldsymbol{d}$ are orthonormal base vectors (directors) in the reference and deformed placement, respectively.

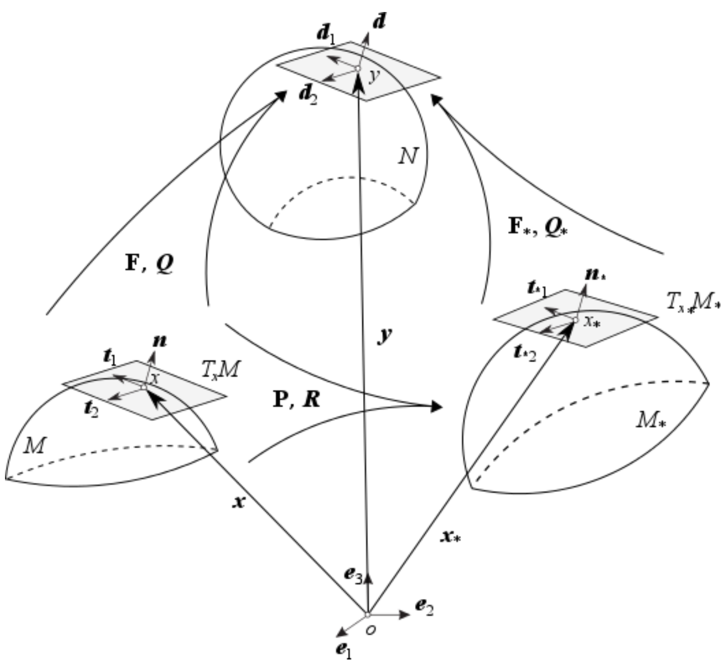

Figure 1. Deformation of the base surface and change of the reference placement

The 2D shell strain and bending measures $\boldsymbol{E}, \boldsymbol{K} \in V \otimes T_x M$ corresponding to the kinematics (2) can be defined according to Chróścielewski et al. (2004) and Pietraszkiewicz et al. (2006) by

$$\boldsymbol{E} = \boldsymbol{J}\mathbf{F} - \boldsymbol{Q}\boldsymbol{I}\,, \quad \boldsymbol{K} = \boldsymbol{C}\mathbf{F} - \boldsymbol{Q}\boldsymbol{B}\,, \tag{3}$$

where $\boldsymbol{I} \in V \otimes T_x M$ and $\boldsymbol{J} \in V \otimes T_y N$ are the inclusion operators at $x \in M$ and $y \in N$, see Gurtin and Murdoch (1975), $\boldsymbol{B} \in V \otimes T_x M$ and $\boldsymbol{C} \in V \otimes T_y N$ are the structure curvature tensors of the shell in the reference and actual placement, $\mathbf{F} \in T_y N \otimes T_x M$ is the tangential surface deformation gradient such that $dy = \mathbf{F}dx$, $\boldsymbol{F} = \boldsymbol{J}\mathbf{F}$, $T_x M$ and $T_y N$ are tangent spaces to M at $x \in M$ and to N at $y \in N$, respectively, $\otimes$ is the tensor product, and $\boldsymbol{F} = Grad_s \boldsymbol{y} \in V \otimes T_x M$ is the surface deformation gradient defined as in Gurtin and Murdoch (1975). The two tensors $\boldsymbol{I}, \boldsymbol{B}$ are basic measures of local geometry of the reference shell base surface M .

In what follows it is more convenient to use the 2D shell strain measures in the material representation

$$\mathbf{E} = \boldsymbol{Q}^T \boldsymbol{E}\,, \quad \mathbf{K} = \boldsymbol{Q}^T \boldsymbol{K}\,. \tag{4}$$

Elastic shells are characterised by the 2D strain energy density $W_\kappa(\mathbf{E}, \mathbf{K})$, per unit area of M, such that the constitutive equations have the form

$$\mathbf{N} = \frac{\partial W_\kappa}{\partial \mathbf{E}}\,, \quad \mathbf{M} = \frac{\partial W_\kappa}{\partial \mathbf{K}}\,, \tag{5}$$

where $\mathbf{N}, \mathbf{M} \in V \otimes T_x M$ are the surface stress resultant and stress couple tensors in the material representation. The function W_κ is usually formulated with regard to the undeformed reference placement κ, so that it depends also on the local geometry of M , that is on $\boldsymbol{I}$ and $\boldsymbol{B}$. Dependence of W_κ on $\boldsymbol{I}$ is trivial and does not require any further discussion. But the dependence of W_κ on $\boldsymbol{B}$ may have a considerable influence on the form of constitutive equations discussed below. This fact is indicated explicitly by writing $W_\kappa \equiv W = W(\mathbf{E}, \mathbf{K}; \boldsymbol{B})$.

Change of Reference Placement

Let κ_* be another reference placement of the shell consisting of the base surface M_* with the position vector $\boldsymbol{x}_*$ in the same inertial frame $(O, \mathbf{e}_i)$, see Fig. 1. Let $\mathbf{P} \in T_{x_*} M_* \otimes T_x M$, $\det \mathbf{P} \neq 0$, be the tangential surface deformation gradient such that $dx_* = \mathbf{P}dx$, and $\boldsymbol{R} \in Orth$ be the orthogonal tensor, $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$, $\det \boldsymbol{R} = \pm 1$, transforming $\boldsymbol{t}_\alpha, \boldsymbol{n}$ into $\boldsymbol{t}_{*\alpha} = \boldsymbol{R}\boldsymbol{t}_\alpha$ and $\boldsymbol{n}_* = \boldsymbol{R}\boldsymbol{n}$, where $\boldsymbol{t}_{*\alpha}, \boldsymbol{n}_*$ is the orthonormal basis on M_*. It is apparent that $\mathbf{P}$ satisfies $\mathbf{P}\boldsymbol{I}^T\boldsymbol{n}=\boldsymbol{n}_*\boldsymbol{I}_*\mathbf{P}=\mathbf{0}$. Since $dy=\mathbf{F}dx=\mathbf{F}_*dx_*$, from the above relations one obtains

$$\mathbf{F}=\mathbf{F}_*\mathbf{P}\,,\quad \boldsymbol{Q}=\boldsymbol{Q}_*\boldsymbol{R}\,. \tag{6}$$

Let $\boldsymbol{B}_*$ be the structure curvature tensor of M_* and $\mathbf{E}_*,\mathbf{K}_*$ be the corresponding 2D shell strain measures in the material representation formulated with regard to M_*. These tensors were expressed by Eremeyev and Pietraszkiewicz (2006) in terms of $\boldsymbol{B},\mathbf{E},\mathbf{K}$ and some 2nd-order tensors describing the change of reference placement as

$$\begin{aligned}&\boldsymbol{B}_*=(\det\boldsymbol{R})\boldsymbol{R}\boldsymbol{B}\mathbf{P}^{-1}-\boldsymbol{L}\,,\\ &\mathbf{E}_*=\boldsymbol{R}\mathbf{E}\mathbf{P}^{-1}+\boldsymbol{R}\boldsymbol{I}\mathbf{P}^{-1}-\boldsymbol{I}\,,\quad \mathbf{K}_*=(\det\boldsymbol{R})\boldsymbol{R}\mathbf{K}\mathbf{P}^{-1}+\boldsymbol{L}\,,\end{aligned} \tag{7}$$

where

$$\boldsymbol{L}=\boldsymbol{R}\boldsymbol{G}\mathbf{P}^{-1}\,,\quad \boldsymbol{G}d\boldsymbol{x}=\operatorname{ax}(d\boldsymbol{R}^T\boldsymbol{R})\,, \tag{8}$$

and $\operatorname{ax}(.)$ is the axial vector of the skew tensor $(.)$. The tensors $\boldsymbol{L},\boldsymbol{G}\in V\otimes T_xM$ describe changes of the structure curvature tensor under change of the reference placement.

Material Symmetry Group

The form of 2D strain energy density $W_*\left(\mathbf{E}_*,\mathbf{K}_*;\boldsymbol{B}_*\right)$ may, in general, be different than that of $W\left(\mathbf{E},\mathbf{K};\boldsymbol{B}\right)$. However, for any part $\Pi_*\subset M_*$ corresponding to $\Pi\subset M$ the elastic strain energy should be conserved,

$$\iint_{\Pi}Wda=\iint_{\Pi_*}W_*da_*\,,\quad da_*=J(\mathbf{P})da\,, \tag{9}$$

because the functions W and W_* describe the strain energy density of the same deformed state. Thus, on physical ground, the elementary surface element da of M should also be preserved during change of the reference placement, i.e. $J(\mathbf{P})=1$. As a result, expectation that W be insensitive to change of the reference placement leads to the following local invariance requirement:

$$W\left(\mathbf{E},\mathbf{K};\boldsymbol{B}\right)=W\left(\boldsymbol{R}\mathbf{E}\mathbf{P}^{-1}+\boldsymbol{R}\boldsymbol{I}\mathbf{P}^{-1}-\boldsymbol{I}\,,(\det\boldsymbol{R})\boldsymbol{R}\mathbf{K}\mathbf{P}^{-1}+\boldsymbol{L};\,(\det\boldsymbol{R})\boldsymbol{R}\boldsymbol{B}\mathbf{P}^{-1}-\boldsymbol{L}\right). \tag{10}$$

The requirement is described by three tensor fields: $\mathbf{P},\boldsymbol{R},\boldsymbol{L}$.

The material symmetry group $\mathcal{G}_\kappa$ consists of all sets of ordered triple of tensors $\mathrm{X}=\left(\mathbf{P},\boldsymbol{R},\boldsymbol{L}\right)$ satisfying the relation (10) for any tensors $\mathbf{E},\mathbf{K},\boldsymbol{B}$ in the domain of definition of the function W. The set $\mathcal{G}_\kappa$ is the group with regard to the group operation $\circ$ defined by

$$\mathrm{X}_1\circ\mathrm{X}_2=\left[\mathbf{P}_1\mathbf{P}_2\,,\boldsymbol{R}_1\boldsymbol{R}_2\,,\boldsymbol{L}_1+(\det\boldsymbol{R}_1)\boldsymbol{R}_1\boldsymbol{L}_2\mathbf{P}_1^{-1}\right], \tag{11}$$

where $\mathrm{X}_1=\left(\mathbf{P}_1,\boldsymbol{R}_1,\boldsymbol{L}_1\right)$ and $\mathrm{X}_2=\left(\mathbf{P}_2,\boldsymbol{R}_2,\boldsymbol{L}_2\right)$ are two different elements of $\mathcal{G}_\kappa$.

It is easy to prove (see Eremeyev and Pietraszkiewicz 2006) that if both $\mathrm{X}_1,\mathrm{X}_2\in\mathcal{G}_\kappa$ then also $\mathrm{X}_1\circ\mathrm{X}_2\in\mathcal{G}_\kappa$, and there exists such X^{-1} that $\mathrm{X}\circ\mathrm{X}^{-1}=\left(\mathbf{A},\mathbf{1},\mathbf{0}\right)\equiv\mathrm{I}$, where $\mathbf{A}=\mathbf{1}-\boldsymbol{n}\otimes\boldsymbol{n}$ is the metric tensor of M, $\mathbf{1}$ is the 3D unit tensor, and I is the unit element of $\mathcal{G}_\kappa$. Moreover, if κ_1 and κ_2 are two different reference placements of the shell having the common tangent plane at the point $x_0 \in M$, in which the symmetry group is discussed, then under change of the reference placement the corresponding material symmetry groups $\mathcal{G}_{\kappa_1}$ and $\mathcal{G}_{\kappa_2}$ transform according to the rule

$$\mathcal{G}_{\kappa_2} = \mathbb{Q} \circ \mathcal{G}_{\kappa_1} \circ \mathbb{Q}^{-1}, \quad \mathbb{Q} \equiv \left(\boldsymbol{R}, \mathbf{P}, \boldsymbol{L}\right). \tag{12}$$

The transformation (12) is a counterpart in the resultant theory of elastic shells of the well known Noll rule for material symmetry groups of simple materials in 3D continuum mechanics, see Truesdell and Noll (1965).

The structure of material symmetry group $\mathcal{G}_\kappa$ puts some constraints on the form of W which allow one to considerably simplify this form. For example, it was proved by Eremeyev and Pietraszkiewicz (2006) that:

1. If $\mathcal{G}_\kappa = \left(\mathbf{A}, \pm \boldsymbol{I}, \mathbf{0}\right)$ then W should be an even function of $\mathbf{K}$ and $\boldsymbol{B}$,
$$W\left(\mathbf{E}, \mathbf{K}; \boldsymbol{B}\right) = W\left(\mathbf{E}, -\mathbf{K}; -\boldsymbol{B}\right). \tag{13}$$
2. If $\mathcal{G}_\kappa$ contains the non-vanishing $\boldsymbol{L}$ then W can be reduced to
$$W\left(\mathbf{E}, \mathbf{K}; \boldsymbol{B}\right) = W\left(\mathbf{E}, \mathbf{K} + \boldsymbol{B}; \mathbf{0}\right). \tag{14}$$
3. If, in addition, explicit dependence of W upon $\boldsymbol{B}$ is neglected, that is when $W = W\left(\mathbf{E}, \mathbf{K}\right)$, then the strain energy function can be reduced to $W = W\left(\mathbf{E}\right)$. This form of W corresponds to the constitutive equation $\mathbf{N} = \partial W / \partial \mathbf{E}$ of the Cosserat membrane, for which $\mathbf{M}$ vanishes and $\mathbf{N}$ is not symmetric, in general.
4. If in $\mathcal{G}_\kappa$ the tensor $\boldsymbol{R}$ is limited to the rotation, $\boldsymbol{R} \in Orth^+$, the strain energy function can be reduced to $W = W\left(\mathbf{Y}\right)$, where $\mathbf{Y} = \sqrt{\boldsymbol{F}^T \boldsymbol{F}}$. This form of W corresponds to the constitutive equation $\mathbf{N} = \partial W / \partial \mathbf{Y}$ of the usual membrane, for which $\mathbf{N}$ is symmetric, $\mathbf{N} = \mathbf{N}^T$, and $\mathbf{M}$ vanishes.
5. If in $\mathcal{G}_\kappa$ the tensor $\mathbf{P}$ is limited to be unimodular, satisfying $\boldsymbol{n}\boldsymbol{I}\mathbf{P} = \mathbf{P}\boldsymbol{I}^T\boldsymbol{n}$, $J(\mathbf{P}) = 1$, it means that W is insensitive to any changes of the reference placement. In such case W can be reduced to $W = \left(\det \mathbf{F}, \boldsymbol{C}\right)$, which is an isotropic function with regard to $\boldsymbol{C}$. Such W describes mechanical properties of the micropolar elastic fluid shells.

Further special cases of $\mathcal{G}_\kappa$ and corresponding simplified forms of constitutive equations, also obtained within other shell models, are discussed in Wang (1973), Gurtin and Murdoch (1975), Altenbach and Zhilin (1988), Steigmann (2001), Eremeyev and Zubov (2003), and Eremeyev (2005).

Solid Elastic Shells

The shell is called *solid* if there exists a reference placement, called undistorted, such that the material symmetry group is given by

$$\mathcal{G}_\kappa = \left\{\mathbf{P} = \boldsymbol{O}\boldsymbol{I}\mathbf{A}, \boldsymbol{O}, \mathbf{0}; \quad \boldsymbol{O} \in S_n \subset Orth_n\right\}, \tag{15}$$

where $Orth_n$ means the group of rotations about $\boldsymbol{n}$ and reflections with regard to the tangent plane $T_x M$. In the case of solid shells, all transformations of the reference placement are assumed to be performed by the same rotation of shell cross sections and of shell directors. Also values of the tangential deformation gradient of transformations of the reference placement are restricted to 2D rotation tensors following from projection of $Orth_n$ onto the tangent plane T_xM. As a result, the group $\mathcal{G}_\kappa$ is described entirely by the subgroup S_n.

For an arbitrary tensor $\boldsymbol{L}$ such that $\boldsymbol{L}\boldsymbol{I}^T\boldsymbol{n}=\mathbf{0}$ and for any $\boldsymbol{O}\subset Orth_n$ the relation $\boldsymbol{L}\boldsymbol{I}^T\boldsymbol{O}\boldsymbol{I}\mathbf{A}=\boldsymbol{L}\boldsymbol{I}^T\boldsymbol{O}\boldsymbol{I}$ is identically satisfied. The invariance requirement for W leads then to establishing all $\boldsymbol{O}\in S_n\subset Orth_n$ such that

$$W(\mathbf{E},\mathbf{K};\boldsymbol{B})=W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T,\,(\det\boldsymbol{O})\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T,\,(\det\boldsymbol{O})\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right]. \tag{16}$$

In the most restricted case of (16), the shell material is insensitive to rotations of shell material elements by an arbitrary angle about $\boldsymbol{n}$ and to reflections with regard to the tangent plane T_xM. Such case can be formally defined as follows:

The solid elastic shell is called *isotropic* if there exist a reference placement, called undistorted, such that the strain energy density satisfies the relation

$$W(\mathbf{E},\mathbf{K};\boldsymbol{B})=W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T,\,(\det\boldsymbol{O})\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T,\,(\det\boldsymbol{O})\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right]\ \text{for any } \boldsymbol{O}\in Orth_n\,. \tag{17}$$

The solid elastic shell is called *hemitropic* if there exist a reference placement, called undistorted, such that the strain energy density satisfies the relation

$$W(\mathbf{E},\mathbf{K};\boldsymbol{B})=W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T,\,\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T,\,\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right]\ \text{for any } \boldsymbol{O}\in Orth_n^+\,, \tag{18}$$

where $Orth_n^+$ is the group of rotations about $\boldsymbol{n}$.

The solid elastic shell is called *orthotropic* if the strain energy density for some reference placement satisfies the relation

$$W(\mathbf{E},\mathbf{K};\boldsymbol{B})=W\left[\boldsymbol{O}\mathbf{E}\boldsymbol{I}^T\boldsymbol{O}^T,\,(\det\boldsymbol{O})\boldsymbol{O}\mathbf{K}\boldsymbol{I}^T\boldsymbol{O}^T,\,(\det\boldsymbol{O})\boldsymbol{O}\boldsymbol{B}\boldsymbol{I}^T\boldsymbol{O}^T\right] \tag{19}$$

for any $\boldsymbol{O}\in S_n=(\mathbf{1},-\mathbf{1},2\boldsymbol{n}\otimes\boldsymbol{n}-\mathbf{1})$. In this case S_n consists of orthogonal tensors performing rotations of 180° about the normal $\boldsymbol{n}$ and reflections with regard to the tangent plane T_xM.

Definitions (17)-(19) are less restrictive than those of isotropic, hemitropic, and orthotropic materials used in 3D non-linear micropolar elasticity, see Eremeyev and Pietraszkiewicz (2012, 2016). For example, in isotropic shells the group $\mathcal{G}_\kappa$ contains the narrower class of orthogonal tensors $Orth_n$ than in the corresponding 3D continuum, where the full orthogonal group $Orth$ is present. As a result, representation theorems applied to W in (17) should lead to more polynomial terms expressed by joint invariants of $\mathbf{E},\mathbf{K},\boldsymbol{B}$ than in the corresponding 3D case. Since these 2nd-order tensors are of mixed type $\in V\otimes T_xM$, not $\in V\otimes V$ as in the 3D case, representation theorems known for 2nd-order tensors $\in V\otimes V$ require an additional non-trivial adjustment. It is fair to state that explicit general forms of W for shells expressed in terms of joint invariants of $\mathbf{E},\mathbf{K},\boldsymbol{B}\in V\otimes T_xM$ for various material symmetries are still waiting for solution.

Physically Linear Elastic shells

In the simplest case of an isotropic elastic shell, for which W still satisfies the relation (17), the strain energy function takes the quadratic form

$$\begin{aligned} 2W &= \alpha_1 \operatorname{tr}^2\mathbf{E}_{||} + \alpha_2 \operatorname{tr}\mathbf{E}_{||}^2 + \alpha_3 \operatorname{tr}\left(\mathbf{E}_{||}^T\mathbf{E}\right) + \alpha_4 \boldsymbol{n}\mathbf{E}\mathbf{E}^T\boldsymbol{n} \\ &+ \beta_1 \operatorname{tr}^2\mathbf{K}_{||} + \beta_2 \operatorname{tr}\mathbf{K}_{||}^2 + \beta_3 \operatorname{tr}\left(\mathbf{K}_{||}^T\mathbf{K}\right) + \beta_4 \boldsymbol{n}\mathbf{K}\mathbf{K}^T\boldsymbol{n}, \end{aligned} \tag{20}$$

where

$$\mathbf{E}_{||} = \mathbf{E} - \boldsymbol{n}\mathbf{E}, \quad \mathbf{K}_{||} = \mathbf{K} - \boldsymbol{n}\mathbf{K}, \tag{21}$$

and W is assumed not to depend explicitly upon $\boldsymbol{B}$. The bilinear terms of $\mathbf{E}$ and $\mathbf{K}$ are not present in (20), because $\mathbf{K}$ is the pseudo-tensor which changes its sign under inversion transformation with regard to $\boldsymbol{n}$. In the expression (20) there are eight scalar factors α_k, β_k, $k = 1,2,3,4$, which can depend on $\boldsymbol{B}$, in general.

The quadratic function (20) generates, according to (5), the following 2D constitutive equations:

$$\begin{aligned} \mathbf{N} &= \alpha_1 \mathbf{A}\operatorname{tr}\mathbf{E}_{||} + \alpha_2 \operatorname{tr}\mathbf{E}_{||}^T + \alpha_3 \mathbf{E} + \alpha_4 \boldsymbol{n}\mathbf{E}\boldsymbol{I}^T\boldsymbol{n}, \\ \mathbf{M} &= \beta_1 \mathbf{A}\operatorname{tr}\mathbf{K}_{||} + \beta_2 \operatorname{tr}\mathbf{K}_{||}^T + \beta_3 \mathbf{K} + \beta_4 \boldsymbol{n}\mathbf{K}\boldsymbol{I}^T\boldsymbol{n}. \end{aligned} \tag{22}$$

A particular case of (20) is the strain energy density of the form

$$\begin{aligned} 2W &= C\left[\nu \operatorname{tr}^2\mathbf{E}_{||} + (1-\nu)\operatorname{tr}\left(\mathbf{E}_{||}^T\mathbf{E}\right)\right] + \alpha_s C(1-\nu)\boldsymbol{n}\mathbf{E}\mathbf{E}^T\boldsymbol{n} \\ &+ D\left[\nu \operatorname{tr}^2\mathbf{K}_{||} + (1-\nu)\operatorname{tr}\left(\mathbf{K}_{||}^T\mathbf{K}\right)\right] + \alpha_t D(1-\nu)\boldsymbol{n}\mathbf{K}\mathbf{K}^T\boldsymbol{n}, \end{aligned} \tag{23}$$

where $C, D, \nu, \alpha_s, \alpha_t$ are independent material constants.

The form (23) of the quadratic strain energy function and corresponding 2D constitutive equations were proposed by Chróścielewski et al. (1992) and used in a number of papers including the book by Chróścielewski et al. (2004). Some refinement of (23) for shells based on the consistent second approximation to the complementary energy density of the geometrically non-linear isotropic elastic shells was proposed by Pietraszkiewicz and Konopińska (2014), see also Pietraszkiewicz (2018).

In more general cases of shell anisotropy, the 2D quadratic strain energy function W and corresponding constitutive equations may require much more terms, particularly when W could be allowed to depend explicitly upon the structure curvature tensor $\boldsymbol{B}$. Some such more complex quadratic functions are discussed in Eremeyev and Pietraszkiewicz (2006).

Cross References

Surface Geometry, Elements
Elastic Shells, Resultant Non-linear Theory
Junctions in Irregular Shell Structures
Shell Thermomechanics, Resultant Non-linear Theory
Cosserat Media

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