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Junctions in Irregular Shell Structures

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Synonyms
Connections of shell elements; Shell branchings and intersections

Definition
Junctions in shells are design elements used for assembling regular shell parts along some of their boundaries into the complex multi-shell structure. The junctions can be constructed as bolted, welded, riveted, glued, adhesively bonded etc. Within the shell model mechanical properties of the junction are characterized by a scalar function of bending measure of the junction curve. This function has to be established by experiments.

Introduction

It follows from the review by Pietraszkiewicz and Konopińska (2015) that different shell models available in the literature require special forms of jump conditions at the singular surface curves modelling the shell junctions. Jump conditions corresponding to different shell models may lead to different stress and strain distributions near the junction. But the review also indicates that almost in all descriptions of shell junctions available in the literature the stiff junction conditions were enforced. Deformability of the junction itself was explicitly indicated and used in only a few papers based on simplest shell models.

Within the resultant non-linear six-field shell model, the mechanical theory of compound multi-shell structures was initiated by Makowski and Stumpf (1994) and developed in the book by Chróścielewski et al. (2004). In this approach several regular shell elements may be joined at the common junction, deformability of any shell branch at the junction may individually be defined, and the junction curve itself may be equipped with additional mechanical properties independent from the adjacent shell branches. Unfortunately, the BVP of such a general irregular shell model becomes extremely complex and virtually useless for engineering applications. Even relatively simple cases of branching and self-intersecting shells developed in Konopińska and Pietraszkiewicz (2007) and Pietraszkiewicz and Konopińska (2011) led to complex shell relations which were still hardly readable for engineering community.

In this entry, following Pietraszkiewicz (2016), we formulate the boundary value equilibrium problem for the simplest compound shell structure consisting of two regular shell elements connected together along the common edge. Particular attention is paid to the static jump conditions across the junction. By further constraining the junction behaviour the stiff junction, the hinge junction, and the deformable junction are described and the corresponding reduced forms of the principle of virtual displacements are formulated.

Notation and Shell Equilibrium Conditions

A shell is a 3D solid body identified in the undeformed placement with a region B of the physical space $\mathcal{E}$ having the translation vector space V. The position vectors $\mathbf{x}$ and $\mathbf{y}=\chi(\mathbf{x})$ relative to some origin $\mathrm{o}\in\mathcal{E}$ of any material particle in the undeformed and deformed placement, respectively, are represented by

$$\mathbf{x}=\boldsymbol{x}+\xi\boldsymbol{t}\,,\quad \mathbf{y}=\boldsymbol{y}(\boldsymbol{x})+\boldsymbol{\zeta}(\boldsymbol{x},\xi)\,,\quad \boldsymbol{\zeta}(\boldsymbol{x},0)=\mathbf{0}\,. \tag{1}$$

Here $\boldsymbol{x}$ and $\boldsymbol{y}$ are position vectors of some shell base surfaces M and $N=\chi(M)$ in the undeformed and deformed placement, respectively, $\boldsymbol{\zeta}$ is a deviation vector from N, $\boldsymbol{n}$ is the unit vector normal to M and orienting it, $\boldsymbol{t}$ is the unit vector not necessarily normal to M with $\boldsymbol{t}\cdot\boldsymbol{n}>0$, and $\xi\in[-h^-,h^+]$ is the distance from M along $\boldsymbol{t}$ with $h=h^-+h^+$ the initial shell thickness measured along ξ.

Within the resultant non-linear shell model (Libai and Simmonds 1998; Chróścielewski et al. 2004; Pietraszkiewicz et al. 2006; Pietraszkiewicz 2018) the 2D non-linear equilibrium conditions in M are derived by the *exact* through-the-thickness integration of 3D equilibrium conditions of continuum mechanics.

Let $\boldsymbol{f},\boldsymbol{c}$ be the external resultant surface force and couple vector fields acting on $\chi(M)$, but measured per unit area od M, and $\boldsymbol{n}^*,\boldsymbol{m}^*$ be the external resultant boundary force and couple vectors prescribed along $\partial N_f=\chi(\partial M_f)$, but measured per unit length of ∂M_f having $\boldsymbol{\nu}$ as the outward unit normal vector. Then for any part $\Pi\subset M$ the 3D equilibrium equations can be reduced to the following 2D resultant equilibrium equations of forces and couples (Pietraszkiewicz 2018)

$$\begin{aligned}&\iint_{\Pi}\boldsymbol{f}\,da+\int_{\partial\Pi\backslash\partial M_f}\boldsymbol{n}_\nu\,ds+\int_{\partial\Pi\cap\partial M_f}\boldsymbol{n}^*\,ds=\mathbf{0}\,,\\&\iint_{\Pi}\boldsymbol{c}\,da+\int_{\partial\Pi\backslash\partial M_f}\left(\boldsymbol{m}_\nu+\boldsymbol{y}\times\boldsymbol{n}_\nu\right)ds+\int_{\partial\Pi\cap\partial M_f}\left(\boldsymbol{m}^*+\boldsymbol{y}\times\boldsymbol{n}^*\right)ds=\mathbf{0}\,,\end{aligned} \tag{2}$$

where

$$\boldsymbol{n}_\nu=\int_{-h^-}^{+h^+}\mathbf{P}\mathbf{n}^*\mu\mathrm{d}\xi=\boldsymbol{n}^\alpha\nu_\alpha\,,\quad \boldsymbol{m}_\nu=\int_{-h^-}^{+h^+}\boldsymbol{\zeta}\times\mathbf{P}\mathbf{n}^*\mu\mathrm{d}\xi=\boldsymbol{m}^\alpha\nu_\alpha\,, \tag{3}$$

$\mathbf{P}$ is the Piola stress tensor in the shell space, $\mathbf{n}^*$ is the external unit normal vector to the 3D undeformed shell cross section $\partial\mathrm{P}^*\subset\partial\mathrm{B}^*$, μ is the geometric expansion factor, and $\boldsymbol{n}^*,\boldsymbol{m}^*$ are statically equivalent to distribution of the external traction $\mathbf{t}^*$ applied on the lateral shell boundary surface $\partial\mathrm{B}^*_f$.

The shell base surface M may be irregular one, in general, consisting of regular parts $M_1,M_2,...,M_n$ joined together along some parts of their edges. The junction curves form together a net of singular surface curves Γ along which the junction jump (also called continuity) conditions should be satisfied. In case of branching and self-intersecting shells such jump conditions were formulated in Konopińska and Pietraszkiewicz (2007).

In order to keep the junction relations in focus, only two shell elements with regular base surfaces M_1 and M_2 connected together along their common edges coinciding with Γ are discussed here, see Fig. 1.

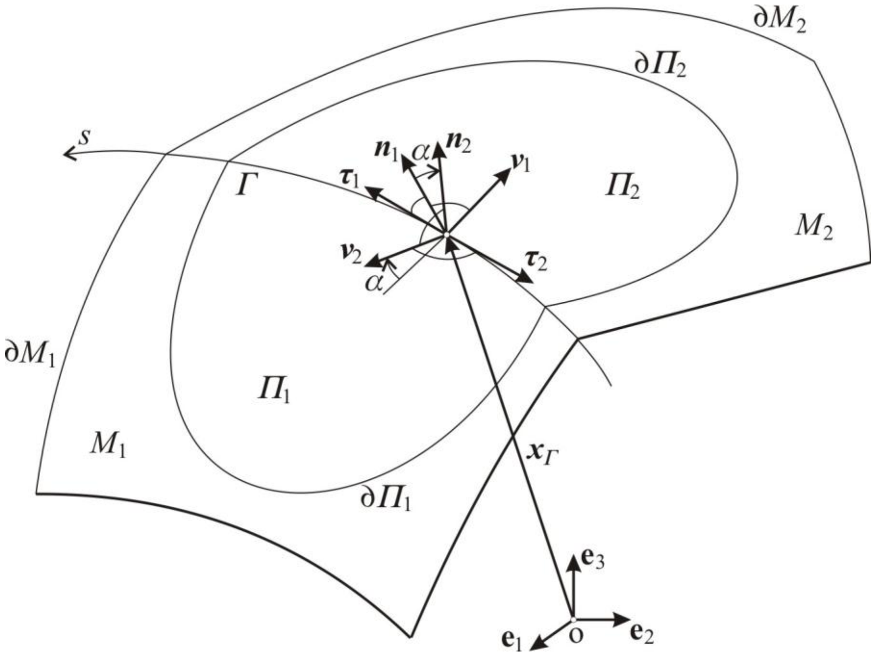

Figure 1. The irregular surface $M = M_1 \cup M_2$ with the fold Γ .

In case of Γ oriented consistently with M_1, at any point $x_\Gamma \in \Gamma$ the unit tangent $\boldsymbol{\tau}_\Gamma = \partial \boldsymbol{x}_\Gamma / \partial s$ of Γ coincides with the unit tangent $\boldsymbol{\tau}_1$ of ∂M_1, $\boldsymbol{\tau}_\Gamma = \boldsymbol{\tau}_1$, and two other unit vectors $\boldsymbol{n}_\Gamma$ and $\boldsymbol{\nu}_\Gamma = \boldsymbol{\tau}_\Gamma \times \boldsymbol{n}_\Gamma$ along Γ coincide with $\boldsymbol{n}_1$ and $\boldsymbol{\nu}_1 = \boldsymbol{\tau}_1 \times \boldsymbol{n}_1$ of ∂M_1, respectively. At the same point $x_\Gamma \in \Gamma$ the orthonormal triad $\boldsymbol{\nu}_2, \boldsymbol{\tau}_2, \boldsymbol{n}_2$ of the edge $\partial \Pi_2 \cap \Gamma \subset \partial M_2$ does not coincide with the triad $\boldsymbol{\nu}_1, \boldsymbol{\tau}_1, \boldsymbol{n}_1$ of $\partial \Pi_1 \cap \Gamma \subset \partial M_1$. This means that the regular surface elements M_1 and M_2 have different orientations and their outward unit normal vectors belong to different 2D tangent spaces.

For any $\Pi \subset M$ having the fold Γ in its interior (Fig. 1), the surface divergence theorems of some terms in (2) are, see Pietraszkiewicz and Konopińska (2011, 2014),

$$\begin{aligned}
&\int_{\partial \Pi} \boldsymbol{n}^\alpha \nu_\alpha \, ds = \iint_{\Pi \setminus \Gamma} \boldsymbol{n}^\alpha |_\alpha \, da - \int_{\Pi \cap \Gamma} [\boldsymbol{n}^\alpha \nu_\alpha] \, ds , \\
&\int_{\partial \Pi} \boldsymbol{m}^\alpha \nu_\alpha \, ds = \iint_{\Pi \setminus \Gamma} \boldsymbol{m}^\alpha |_\alpha \, da - \int_{\Pi \cap \Gamma} [\boldsymbol{m}^\alpha \nu_\alpha] \, ds , \\
&\int_{\partial \Pi} \left(\boldsymbol{y} \times \boldsymbol{n}^\alpha \nu_\alpha \right) ds = \iint_{\Pi \setminus \Gamma} \left(\boldsymbol{y},_\alpha \times \boldsymbol{n}^\alpha + \boldsymbol{y} \times \boldsymbol{n}^\alpha |_\alpha \right) da - \int_{\Pi \cap \Gamma} [\boldsymbol{y} \times \boldsymbol{n}^\alpha \nu_\alpha] ds ,
\end{aligned} \tag{4}$$

where $(.)|_\alpha$ is the covariant surface derivative taken in the undeformed surface metric $a_{\alpha\beta}$ of M , and the jump terms are defined by

$$\begin{aligned}
&[\boldsymbol{n}^\alpha \nu_\alpha] = \boldsymbol{n}_1^\alpha \nu_{1\alpha} + \boldsymbol{n}_2^\alpha \nu_{2\alpha} , \quad [\boldsymbol{m}^\alpha \nu_\alpha] = \boldsymbol{m}_1^\alpha \nu_{1\alpha} + \boldsymbol{m}_2^\alpha \nu_{2\alpha} , \\
&[\boldsymbol{y} \times \boldsymbol{n}^\alpha \nu_\alpha] = \boldsymbol{y}_1 \times \boldsymbol{n}_1^\alpha \nu_{1\alpha} + \boldsymbol{y}_2 \times \boldsymbol{n}_2^\alpha \nu_{2\alpha} .
\end{aligned} \tag{5}$$

With (3)-(5) the resultant equilibrium conditions (2) are equivalent to the local equilibrium equations satisfied for any part $\Pi \setminus \Gamma \subset M$,

$$\boldsymbol{n}^{\alpha}|_{\alpha} + \boldsymbol{f} = \boldsymbol{0}, \quad \boldsymbol{m}^{\alpha}|_{\alpha} + \boldsymbol{y},_{\alpha} \times \boldsymbol{n}^{\alpha} + \boldsymbol{c} = \boldsymbol{0}, \tag{6}$$

the natural static boundary conditions satisfied along ∂M_f,

$$\boldsymbol{n}^{\alpha}\nu_{\alpha} - \boldsymbol{n}^{*} = \boldsymbol{0}, \quad \boldsymbol{m}^{\alpha}\nu_{\alpha} - \boldsymbol{m}^{*} = \boldsymbol{0}, \tag{7}$$

and the static jump conditions across Γ,

$$[\boldsymbol{n}^{\alpha}\nu_{\alpha}] = \boldsymbol{0}, \quad [\boldsymbol{m}^{\alpha}\nu_{\alpha}] + [\boldsymbol{y} \times \boldsymbol{n}^{\alpha}\nu_{\alpha}] = \boldsymbol{0}. \tag{8}$$

The deformed position vector field $\boldsymbol{y}$ is additionally assumed here to be always smooth, so that $[\boldsymbol{y}] = \boldsymbol{0}$ across Γ. By this requirement the shell is prevented from decomposing along Γ during deformation. As a result, the static jump conditions (8) are reduced to

$$[\boldsymbol{n}^{\alpha}\nu_{\alpha}] = \boldsymbol{0}, \quad [\boldsymbol{m}^{\alpha}\nu_{\alpha}] = \boldsymbol{0}. \tag{9}$$

If Γ is oriented consistently with M_1, then (Fig. 1)

$$\boldsymbol{\tau}_2 = -\boldsymbol{\tau}_1, \quad \boldsymbol{n}_2 = \boldsymbol{n}_1 \cos\alpha + \boldsymbol{\nu}_1 \sin\alpha, \quad \boldsymbol{\nu}_2 = -\boldsymbol{\nu}_1 \cos\alpha + \boldsymbol{n}_1 \sin\alpha. \tag{10}$$

Hence, in this case the static jumps across Γ are defined as follows:

$$[\boldsymbol{n}^{\alpha}\nu_{\alpha}] = -\left(\boldsymbol{n}_2 \cos\alpha - \boldsymbol{n}_1\right)\nu_1, \quad [\boldsymbol{m}^{\alpha}\nu_{\alpha}] = -\left(\boldsymbol{m}_2 \cos\alpha - \boldsymbol{m}_1\right)\nu_1. \tag{11}$$

Principle of Virtual Displacements

Let $(\boldsymbol{v}, \boldsymbol{w}) \in V$ be two vector fields smooth at the regular points of $M \setminus \Gamma$, and $(\boldsymbol{v}_{\Gamma}, \boldsymbol{w}_{\Gamma}) \in V$ be two other vector fields smooth along Γ. Then for any part $\Pi \subset M$ containing the fold Γ one can set the integral identity

$$\begin{aligned}
&\iint_{\Pi\setminus\Gamma} \left\{ \left(\boldsymbol{n}^{\alpha}|_{\alpha} + \boldsymbol{f} \right) \cdot \boldsymbol{v} + \left(\boldsymbol{m}^{\alpha}|_{\alpha} + \boldsymbol{y},_{\alpha} \times \boldsymbol{n}^{\alpha} + \boldsymbol{c} \right) \cdot \boldsymbol{w} \right\} da \\
&\quad + \int_{\Pi\cap\partial M_f} \left\{ \left(\boldsymbol{n}^{*} - \boldsymbol{n}^{\alpha}\nu_{\alpha} \right) \cdot \boldsymbol{v} + \left(\boldsymbol{m}^{*} - \boldsymbol{m}^{\alpha}\nu_{\alpha} \right) \cdot \boldsymbol{w} \right\} ds \\
&\quad - \int_{\Pi\cap\Gamma} \left\{ [\boldsymbol{n}^{\alpha}\nu_{\alpha}] \cdot \boldsymbol{v}_{\Gamma} + [\boldsymbol{m}^{\alpha}\nu_{\alpha}] \cdot \boldsymbol{w}_{\Gamma} \right\} ds = 0.
\end{aligned} \tag{12}$$

Since

$$\begin{aligned}
&\boldsymbol{n}^{\alpha}|_{\alpha} \cdot \boldsymbol{v} = (\boldsymbol{n}^{\alpha} \cdot \boldsymbol{v})|_{\alpha} - \boldsymbol{n}^{\alpha} \cdot \boldsymbol{v},_{\alpha}, \quad \boldsymbol{m}^{\alpha}|_{\alpha} \cdot \boldsymbol{v} = (\boldsymbol{m}^{\alpha} \cdot \boldsymbol{v})|_{\alpha} - \boldsymbol{m}^{\alpha} \cdot \boldsymbol{v},_{\alpha}, \\
&(\boldsymbol{y},_{\alpha} \times \boldsymbol{n}^{\alpha}) \cdot \boldsymbol{w} = -\boldsymbol{n}^{\alpha} \cdot (\boldsymbol{y},_{\alpha} \times \boldsymbol{w}),
\end{aligned} \tag{13}$$

and Π is an arbitrarily chosen part of M containing Γ, transforming (12) with the help of (13) and applying the surface divergence theorems (4) we obtain

$$\begin{aligned}
&-\iint_{\Pi\setminus\Gamma} \left\{ \boldsymbol{n}^{\alpha} \cdot \left(\boldsymbol{v},_{\alpha} + \boldsymbol{y},_{\alpha} \times \boldsymbol{w} \right) + \boldsymbol{m}^{\alpha} \cdot \boldsymbol{w},_{\alpha} \right\} da + \iint_{\Pi\setminus\Gamma} \left(\boldsymbol{f} \cdot \boldsymbol{v} + \boldsymbol{c} \cdot \boldsymbol{w} \right) da \\
&\quad + \int_{\Pi\cap\partial M_f} \left(\boldsymbol{n}^{*} \cdot \boldsymbol{v} + \boldsymbol{m}^{*} \cdot \boldsymbol{w} \right) ds + \int_{\Pi\cap\partial M_d} \left(\boldsymbol{n}^{\alpha}\nu_{\alpha} \cdot \boldsymbol{v} + \boldsymbol{m}^{\alpha}\nu_{\alpha} \cdot \boldsymbol{w} \right) ds \\
&\quad + \int_{\Pi\cap\Gamma} \left\{ [\boldsymbol{n}^{\alpha}\nu_{\alpha} \cdot \boldsymbol{v}] - [\boldsymbol{n}^{\alpha}\nu_{\alpha}] \cdot \boldsymbol{v}_{\Gamma} + [\boldsymbol{m}^{\alpha}\nu_{\alpha} \cdot \boldsymbol{w}] - [\boldsymbol{m}^{\alpha}\nu_{\alpha}] \cdot \boldsymbol{w}_{\Gamma} \right\} ds = 0.
\end{aligned} \tag{14}$$

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are interpreted as the kinematically admissible virtual translation and rotation vectors such that $\boldsymbol{v}=\boldsymbol{w}=\boldsymbol{0}$ along ∂M_d, then the integral over $\Pi \cap \partial M_d$ in (14) identically vanishes. The second surface integral over $\Pi \setminus \Gamma$ and the one over $\Pi \cap \partial M_f$ in (14) can then be interpreted as the external virtual work performed by the given surface $\boldsymbol{f},\boldsymbol{c}$ and boundary $\boldsymbol{n}^*,\boldsymbol{m}^*$ loads, respectively. In this context the first surface integral over $\Pi \setminus \Gamma$ takes the meaning of internal virtual work, where the expressions $\boldsymbol{v},_\alpha + \boldsymbol{y},_\alpha \times \boldsymbol{w}$ and $\boldsymbol{w},_\alpha$ can be interpreted as just virtual changes of appropriately defined shell strain and bending vectors, respectively. Then the formula (14) takes the meaning of the principle of virtual displacements for the irregular shell structure.

Let the vector field $\boldsymbol{u}(\boldsymbol{x})$ represent the work-averaged translations of M and the proper orthogonal tensor field $\boldsymbol{Q}(\boldsymbol{x})$ represent the work-averaged rotations of the shell cross sections. The deformed shell configuration can then be described by the relations

$$\boldsymbol{y}=\boldsymbol{x}+\boldsymbol{u}, \quad \boldsymbol{d}_\alpha=\boldsymbol{Q}\boldsymbol{t}_\alpha, \quad \boldsymbol{d}=\boldsymbol{Q}\boldsymbol{t}, \tag{15}$$

where $\boldsymbol{t}_\alpha,\boldsymbol{t}$ and $\boldsymbol{d}_\alpha,\boldsymbol{d}$ are three directors attached to any point of the undeformed M and deformed $N=\chi(M)$ base surfaces, respectively.

Let us consider a one-parametric family of shell deformations described by $\boldsymbol{y}(\boldsymbol{x},t)=\boldsymbol{x}+\boldsymbol{u}(\boldsymbol{x},t)$ and $\boldsymbol{Q}(\boldsymbol{x},t)=\boldsymbol{d}^\alpha(\boldsymbol{x},t)\otimes\boldsymbol{d}_\alpha+\boldsymbol{d}(\boldsymbol{x},t)\otimes\boldsymbol{t}$, where t is a scalar (time-like) parameter such that $t=0$ corresponds to the undeformed shell placement and t to the deformed one. Then the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in (14) can be interpreted as virtual changes of $\boldsymbol{u}$ and $\boldsymbol{Q}$ (linear and angular velocities in a real motion) according to

$$\boldsymbol{v}=\frac{\partial \boldsymbol{y}(\boldsymbol{x},t)}{\partial t}\equiv\delta\boldsymbol{u}, \quad \boldsymbol{w}=\operatorname{ax}\left(\dot{\boldsymbol{Q}}(\boldsymbol{x},t)\boldsymbol{Q}\right)=\operatorname{ax}\left(\delta\boldsymbol{Q}\boldsymbol{Q}^T\right)\equiv\boldsymbol{\omega}, \tag{16}$$

Under such identification, the shell strain $\boldsymbol{\varepsilon}_\alpha$ and bending $\boldsymbol{\kappa}_\alpha$ vectors corresponding to the kinematics (15) and their virtual changes can be defined as in Chróścielewski et al. (1992)

$$\begin{aligned} \boldsymbol{\varepsilon}_\alpha &= \boldsymbol{y},_\alpha - \boldsymbol{d}_\alpha = \boldsymbol{u},_\alpha + (\boldsymbol{1}-\boldsymbol{Q})\boldsymbol{t}_\alpha = E_{\alpha\beta}\boldsymbol{d}^\beta + E_\alpha \boldsymbol{d}, \\ \boldsymbol{\kappa}_\alpha &= \operatorname{ax}\left(\boldsymbol{Q},_\alpha \boldsymbol{Q}^T\right) = \frac{1}{2}\left(\boldsymbol{d}^\beta \times \boldsymbol{Q},_\alpha \boldsymbol{Q}^T \boldsymbol{d}_\beta + \boldsymbol{d}\times\boldsymbol{Q},_\alpha\boldsymbol{Q}^T\boldsymbol{d}\right) = \boldsymbol{d}\times K_{\alpha\beta}\boldsymbol{d}^\beta + K_\alpha \boldsymbol{d}, \end{aligned} \tag{17}$$

$$\begin{aligned} \delta^c \boldsymbol{\varepsilon}_\alpha &= \delta\boldsymbol{u},_\alpha + \boldsymbol{y},_\alpha \times \boldsymbol{\omega} = \delta E_{\alpha\beta}\boldsymbol{d}^\beta + \delta E_\alpha \boldsymbol{d}, \\ \delta^c \boldsymbol{\kappa}_\alpha &= \boldsymbol{\omega},_\alpha = \boldsymbol{d}\times\delta K_{\alpha\beta}\boldsymbol{d}^\beta + \delta K_\alpha \boldsymbol{d}, \end{aligned} \tag{18}$$

where $\delta^c(.)=\boldsymbol{Q}\left\{\delta[\boldsymbol{Q}^T(.)]\right\}$ is the co-rotational variation (the co-rotational time derivative in a real motion), and $\boldsymbol{1}\in V\otimes V$ is the metric tensor of the 3D vector space.

The vectors $\boldsymbol{n}^\alpha,\boldsymbol{m}^\alpha$ and $\boldsymbol{f},\boldsymbol{c}$ appearing in (6) can naturally be represented through components relative to the rotated base $\boldsymbol{d}_\beta,\boldsymbol{d}$ by

$$\begin{aligned} \boldsymbol{n}^\alpha &= N^{\alpha\beta}\boldsymbol{d}_\beta + Q^\alpha \boldsymbol{d}, \quad \boldsymbol{m}^\alpha = \boldsymbol{d}\times M^{\alpha\beta}\boldsymbol{d}_\beta + M^\alpha \boldsymbol{d} = \varepsilon_{\lambda\beta}M^{\alpha\lambda}\boldsymbol{d}^\beta + M^\alpha \boldsymbol{d}, \\ \boldsymbol{f} &= f^\beta \boldsymbol{d}_\beta + f\boldsymbol{d}, \quad \boldsymbol{c} = \boldsymbol{d}\times c^\beta \boldsymbol{d}_\beta + c\boldsymbol{d} = \varepsilon_{\lambda\beta}c^\lambda \boldsymbol{d}^\beta + c\boldsymbol{d}. \end{aligned} \tag{19}$$

The components $M^{\alpha} = \boldsymbol{m}^{\alpha} \cdot \boldsymbol{d}$ are usually called the drilling couples, while the work-conjugate components $K_{\alpha} = \boldsymbol{\kappa}_{\alpha} \cdot \boldsymbol{d}$ are the drilling bendings. These surface stress and strain measures do not appear in any other non-linear shell model.

From (15) and (16) it follows that the displacement boundary conditions, which assure vanishing of the integral over $\Pi \cap \partial M_d$ in (14) should be

$$\boldsymbol{u} - \boldsymbol{u}^* = \boldsymbol{0}, \quad \boldsymbol{Q} - \boldsymbol{Q}^* = \boldsymbol{0}, \tag{20}$$

where $\boldsymbol{u}^*, \boldsymbol{Q}^*$ are the prescribed fields.

Introducing the virtual strain energy density in $M \setminus \Gamma$ defined by

$$\sigma = \boldsymbol{n}^{\alpha} \cdot \delta^c \boldsymbol{\varepsilon}_{\alpha} + \boldsymbol{m}^{\alpha} \cdot \delta^c \boldsymbol{\kappa}_{\alpha}, \tag{21}$$

the principle of virtual displacements following from (14) for the irregular shell structure takes the form

$$\begin{aligned} \iint_{M \setminus \Gamma} \sigma \, da &= \iint_{M \setminus \Gamma} \left(\boldsymbol{f} \cdot \delta \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega} \right) da + \int_{\partial M_f} \left(\boldsymbol{n}^* \cdot \delta \boldsymbol{u} + \boldsymbol{m}^* \cdot \boldsymbol{\omega} \right) ds \\ &+ \int_{\Gamma} \left\{ [\boldsymbol{n}^{\alpha} \nu_{\alpha} \cdot \delta \boldsymbol{u}] - [\boldsymbol{n}^{\alpha} \nu_{\alpha}] \cdot \boldsymbol{v}_{\Gamma} + [\boldsymbol{m}^{\alpha} \nu_{\alpha} \cdot \boldsymbol{\omega}] - [\boldsymbol{m}^{\alpha} \nu_{\alpha}] \cdot \boldsymbol{w}_{\Gamma} \right\} ds. \end{aligned} \tag{22}$$

The curvilinear integral over Γ in (22) contains the jump terms which describe the shell – junction interaction between two joined shell elements with regular base surfaces M_1 and M_2. Explicit expressions of the jump terms depend on the type of junction modelled by this approach. The large variety of types of 1D structural elements, which can be used as junctions in compound shell structures, together with complex kinematics required within the resultant six-field shell model, makes the general non-linear BVP of such structures very complex and hardly readable in engineering applications.

The compound jump terms in (22) can be decomposed as follows:

$$\begin{aligned} [\boldsymbol{n}^{\alpha} \nu_{\alpha} \cdot \delta \boldsymbol{u}] &= [\boldsymbol{n}^{\alpha} \nu_{\alpha}] \cdot < \delta \boldsymbol{u} > + < \boldsymbol{n}^{\alpha} \nu_{\alpha} > \cdot [\delta \boldsymbol{u}], \\ [\boldsymbol{m}^{\alpha} \nu_{\alpha} \cdot \boldsymbol{\omega}] &= [\boldsymbol{m}^{\alpha} \nu_{\alpha}] \cdot < \boldsymbol{\omega} > + < \boldsymbol{m}^{\alpha} \nu_{\alpha} > \cdot [\boldsymbol{\omega}], \end{aligned} \tag{23}$$

where $< \boldsymbol{a} >$ is the average value of $\boldsymbol{a} \in V$ at Γ. In our special case of smooth translations everywhere discussed here, the translation at the junction curve Γ may be interpreted as the common translation of both edges $\partial M_1 \cap \Gamma$ and $\partial M_2 \cap \Gamma$, so that $< \delta \boldsymbol{u} >= \delta \boldsymbol{u}_{\Gamma} \equiv \boldsymbol{v}_{\Gamma}$. But the rotation tensors $\boldsymbol{Q}_1 = \boldsymbol{Q}|_{\partial M_1 \cap \Gamma}$ and $\boldsymbol{Q}_2 = \boldsymbol{Q}|_{\partial M_2 \cap \Gamma}$ of the edges at the same $x_{\Gamma} \in \Gamma$ may be different, in general, $\boldsymbol{Q}_1 \neq \boldsymbol{Q}_2$.

With (23) the principle of virtual displacements (22) can be reduced to

$$\begin{aligned} \iint_{M \setminus \Gamma} \sigma \, da &= \iint_{M \setminus \Gamma} \left(\boldsymbol{f} \cdot \delta \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega} \right) da + \int_{\partial M_f} \left(\boldsymbol{n}^* \cdot \delta \boldsymbol{u} + \boldsymbol{m}^* \cdot \boldsymbol{\omega} \right) ds \\ &+ \int_{\Gamma} \left\{ [\boldsymbol{m}^{\alpha} \nu_{\alpha}] \left(< \boldsymbol{\omega} > - \boldsymbol{w}_{\Gamma} \right) + < \boldsymbol{m}^{\alpha} \nu_{\alpha} > \cdot [\boldsymbol{\omega}] \right\} ds. \end{aligned} \tag{24}$$

Let us introduce explicitly the net rotation tensor $\boldsymbol{Q}_{\Gamma}$ of Γ such that $\boldsymbol{Q}_2 = \boldsymbol{Q}_{\Gamma} \boldsymbol{Q}_1$ at any $x_{\Gamma} \in \Gamma$ when x_{Γ} is approached from both sides of Γ, respectively. Since $\boldsymbol{Q}_2, \boldsymbol{Q}_{\Gamma}, \boldsymbol{Q}_1$ are all proper orthogonal tensors, then

$$\boldsymbol{Q}_2\boldsymbol{Q}_2^T=\mathbf{1}\,,\quad \boldsymbol{Q}_\Gamma\boldsymbol{Q}_\Gamma^T=\mathbf{1}\,,\quad \boldsymbol{Q}_1\boldsymbol{Q}_1^T=\mathbf{1}\,. \tag{25}$$

Virtual changes of these orthogonality relations lead to

$$\begin{aligned}\delta\boldsymbol{Q}_2\boldsymbol{Q}_2^T&=-\boldsymbol{Q}_2\delta\boldsymbol{Q}_2^T=\boldsymbol{\omega}_2\times\mathbf{1},\\ \delta\boldsymbol{Q}_\Gamma\boldsymbol{Q}_\Gamma^T&=-\boldsymbol{Q}_\Gamma\delta\boldsymbol{Q}_\Gamma^T=\boldsymbol{\omega}_\Gamma\times\mathbf{1},\\ \delta\boldsymbol{Q}_1\boldsymbol{Q}_1^T&=-\boldsymbol{Q}_1\delta\boldsymbol{Q}_1^T=\boldsymbol{\omega}_1\times\mathbf{1},\end{aligned} \tag{26}$$

$$\boldsymbol{\omega}_2=\boldsymbol{\omega}_\Gamma+\boldsymbol{Q}_\Gamma\boldsymbol{\omega}_1\,. \tag{27}$$

The virtual rotations $\boldsymbol{\omega}_2, \boldsymbol{\omega}_\Gamma$ and $\boldsymbol{\omega}_1$ are all defined in the shell deformed placement.

Let the virtual rotation $\boldsymbol{w}_\Gamma$ at Γ be interpreted in terms of $\boldsymbol{\omega}$ as

$$<\boldsymbol{\omega}>=\frac{1}{2}\left\{\boldsymbol{\omega}_\Gamma+(\boldsymbol{Q}_\Gamma+\mathbf{1})\boldsymbol{\omega}_1\right\}\equiv\boldsymbol{w}_\Gamma\,. \tag{28}$$

Then the principle (24) can be further reduced to

$$\begin{aligned}\iint_{M\backslash\Gamma}\sigma\,da&=\iint_{M\backslash\Gamma}\left(\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{c}\cdot\boldsymbol{\omega}\right)da+\int_{\partial M_f}\left(\boldsymbol{n}^*\cdot\delta\boldsymbol{u}+\boldsymbol{m}^*\cdot\boldsymbol{\omega}\right)ds\\ &+\int_\Gamma<\boldsymbol{m}^\alpha\nu_\alpha>\cdot[\boldsymbol{\omega}]\,ds\,.\end{aligned} \tag{29}$$

The variational statement (29) governs the simplified BVP of two shell elements with regular base surfaces M_1 and M_2 joined along the junction Γ. This statement has been constructed under the assumption that the junction translations are smooth everywhere during deformations. As a result, kinematic description of the junction has been reduced to characterising how the rotations $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ of the neighbouring points of the junction are related to each other during deformation. This still allows one for a variety of possible characterisations of the junction. Some of the simplest particularly appealing junction characterisations are discussed below.

Mechanical Description of Junction

The stiff junction

The shell junction along Γ is called *stiff* if the shell deformation is continuous on the whole $M=M_1\cup M_2$ including Γ. In this case

$$[\delta\boldsymbol{u}]=\mathbf{0}\,,\quad[\boldsymbol{\omega}]=\mathbf{0}\,,\quad\boldsymbol{u}_1=\boldsymbol{u}_2\,,\quad\boldsymbol{Q}_1=\boldsymbol{Q}_2\,, \tag{30}$$

and the curvilinear integral along Γ in (29) vanishes. The correspondingly simplified PVD is reduced to

$$\iint_M\sigma\,da=\iint_M\left(\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{c}\cdot\boldsymbol{\omega}\right)da+\int_{\partial M_f}\left(\boldsymbol{n}^*\cdot\delta\boldsymbol{u}+\boldsymbol{m}^*\cdot\boldsymbol{\omega}\right)ds\,. \tag{31}$$

The physical meaning of (31) is that in this case the junction along Γ does not contribute to the virtual work of the compound shell structure. The mechanical behaviour of the junction itself is enforced by the behaviour of stiffly joined shell lateral boundary surfaces of regular shell parts with surface elements M_1 and M_2. This is exactly the case of almost all types of shell junctions reviewed in Pietraszkiewicz and Konopińska (2015). In particular, within the resultant non-linear six-field shell theory several folded and multi-shell structures with stiff junctions were modelled and analysed with FEM by Chróścielewski et al. (1997). Non-linear dynamic problems of such structures were discussed by Chróścielewski et al. (2002). A number of linear and non-linear FE solutions of multi-shells with stiff junctions was summarized in the book by Chróścielewski et al. (2004).

The Hinge Junction

The *hinge* junction along Γ is understood when $\boldsymbol{u}$ is continuous across Γ, that is $[\delta\boldsymbol{u}]=\boldsymbol{0}$, $\boldsymbol{u}_1=\boldsymbol{u}_2$, but $\boldsymbol{Q}_1,\boldsymbol{Q}_2$ are entirely unconstrained when approaching Γ along paths on corresponding M_1, M_2. In this case $[\boldsymbol{\omega}]\neq\boldsymbol{0}$, in general. However, in order the entirely unconstrained rotations $\boldsymbol{Q}_1,\boldsymbol{Q}_2$ to happen, from equilibrium it follows that no moments at both sides of Γ should be allowed,

$$\boldsymbol{m}_1^{\alpha}\nu_{1\alpha}=\boldsymbol{0}, \quad \boldsymbol{m}_2^{\alpha}\nu_{\alpha 2}=\boldsymbol{0}, \tag{32}$$

so that $<\boldsymbol{m}^{\alpha}\nu_{\alpha}>=\boldsymbol{0}$ and hence $<\boldsymbol{m}^{\alpha}\nu_{\alpha}>\cdot[\boldsymbol{\omega}]=\boldsymbol{0}$ along Γ. As a result, in the corresponding principle (29) the curvilinear integral along Γ vanishes as well reducing it again formally to (31). However, the important difference from the stiff junction is that in the case of hinge junction along Γ the additional static equilibrium conditions (32) have to be enforced in the process of solution of the BVP.

The Deformable Junction

In the principle (29) both ingredients $<\boldsymbol{m}^{\alpha}\nu_{\alpha}>$ and $[\boldsymbol{\omega}]$ in the last integral may not together identically vanish, in general, that is $<\boldsymbol{m}^{\alpha}\nu_{\alpha}>\neq\boldsymbol{0}$ and $[\boldsymbol{\omega}]\neq\boldsymbol{0}$. In this general case the shell junction along Γ may be called *deformable.*

From engineering point of view, the junctions can be classified according to:

- The type of medium used: bolted, welded, riveted, glued, adhesively bonded etc.
- The type of internal forces the junction is expected to transmit: membrane, shear, moment (stiff, deformable).
- The type of elements the junction is joining: regular shell elements, transition stiffening beam, special junction constructions.

This leads to a large variety of constructions of junctions in compound shell structures. Mechanical and/or deformability properties of each particular case of such junction should be known in advance before the analyses take place.

Let us differentiate the orthogonality relations (25) along Γ,

$$\begin{aligned}(\boldsymbol{Q}_2)'\boldsymbol{Q}_2^T &= -\boldsymbol{Q}_2(\boldsymbol{Q}_2^T)' = \boldsymbol{\kappa}_2\times\boldsymbol{1},\\ (\boldsymbol{Q}_\Gamma)'\boldsymbol{Q}_\Gamma^T &= -\boldsymbol{Q}_\Gamma(\boldsymbol{Q}_\Gamma^T)' = \boldsymbol{\kappa}_\Gamma\times\boldsymbol{1},\\ (\boldsymbol{Q}_1)'\boldsymbol{Q}_1^T &= -\boldsymbol{Q}_1(\boldsymbol{Q}_1^T)' = \boldsymbol{\kappa}_1\times\boldsymbol{1}, \quad (.)'=\frac{d}{ds}(.),\end{aligned} \tag{33}$$

$$\boldsymbol{\kappa}_2=\boldsymbol{\kappa}_\Gamma+\boldsymbol{Q}_\Gamma\boldsymbol{\kappa}_1\,. \tag{34}$$

The vector $\boldsymbol{\kappa}_{\Gamma}$ describes the bending properties of the junction curve Γ during shell deformation.

The mechanical behaviour of the deformable junction can be characterized by the relation

$$< \boldsymbol{m}^{\alpha} \nu_{\alpha} > = f(\boldsymbol{\kappa}_{\Gamma}), \tag{35}$$

where f is a smooth vector function of vectorial argument at any $x_{\Gamma} \in \Gamma$. The relation (35) is the kind of 1D constitutive equation modelling deformability properties of real engineering junctions. It is apparent that due to possible complexity of engineering junction constructions the function f should be established from appropriate experiments for each particular type of the junction.

With (35) and (27) the PVD (24) takes the modified form

$$\begin{aligned} \iint_{M\setminus\Gamma} \sigma\, da &= \iint_{M\setminus\Gamma} \left(\boldsymbol{f} \cdot \delta \boldsymbol{u} + \boldsymbol{c} \cdot \boldsymbol{\omega} \right) da + \int_{\partial M_f} \left(\boldsymbol{n}^* \cdot \delta \boldsymbol{u} + \boldsymbol{m}^* \cdot \boldsymbol{\omega} \right) ds \\ &+ \int_{\Gamma} f(\boldsymbol{\kappa}_{\Gamma}) \cdot \left\{ \boldsymbol{\omega}_{\Gamma} + (\boldsymbol{Q}_{\Gamma} - \mathbf{1}) \boldsymbol{\omega}_1 \right\} ds . \end{aligned} \tag{36}$$

The Elastic Junction

If there exists a scalar function $W(\boldsymbol{\kappa}_{\Gamma})$ such that $f(\boldsymbol{\kappa}_{\Gamma}) = \partial W / \partial \boldsymbol{\kappa}_{\Gamma}$, the junction along Γ may be called *elastic*. The function W may be quite complex non-linear function of $\boldsymbol{\kappa}_{\Gamma}$, so that such a junction is *non-linearly elastic,* in general. But in some cases W may become a quadratic function such that

$$W = \frac{1}{2} \boldsymbol{L} : (\boldsymbol{\kappa}_{\Gamma} \otimes \boldsymbol{\kappa}_{\Gamma}), \quad f(\boldsymbol{\kappa}_{\Gamma}) = \boldsymbol{L} \boldsymbol{\kappa}_{\Gamma}, \tag{37}$$

where $\boldsymbol{L}$ is the second- order tensor of rotational material properties along Γ. In this case the shell junction can be called *linearly elastic.*

Within the non-linear theory of thin shells of Kirchhoff-Love type, description of several types of shell junctions were given by Makowski et al. (1998, 1999) and explicit numerical solutions of the shell of revolution with deformable elasto-plastic junctions were given by Chróścielewski et al. (2011a,b).

Cross References

Surface Geometry, Elements
Elastic Shells, Resultant Non-linear Theory
Shell Thermomechanics, Resultant Non-linear Theory

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