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Shell Thermomechanics, Resultant Non-linear Theory

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Synonyms

Rational thermomechanics of shells; Dynamically and thermally exact shell theory

Definition

Shell thermomechanics is the study of effects of heat upon mechanical properties of a thin solid body. The resultant shell theory is based on the set of two-dimensional balance laws of mass, linear and angular momenta, and energy as well as the entropy inequality which are formulated on the shell base surface as exact resultant implications of corresponding laws of three-dimensional rational continuum thermomechanics. The only approximations enter this shell model through constitutive equations, which are experimental laws anyway.

Introduction

Non-linear thermomechanic two-dimensional (2D) models of shells are usually developed using two main approaches: 1) the so-called *direct* formulation, and 2) the *derived* or *deductive* formulation from three-dimensional (3D) continuum thermomechanics. But the final 2D relations of shell thermomechanics and physical interpretation of their ingredients vary substantially throughout the literature.

The resultant shell thermomechanics proposed by Simmonds (1984, 2012) seems to be the most promising way to formulate shell thermomechanics. All 2D relations were formulated on the shell base surface by exact through-the-thickness integration of appropriate 3D fields of rational continuum thermomechanics. The only approximations were made in the resultant balance of energy when expressed through the 2D stress and strain fields alone. The approximations were then transferred onto the resultant entropy inequality and the 2D constitutive equations, which are experimental laws anyway. The mechanical part of such resultant shell theory, originally proposed by Reissner (1974), gained considerable attention in the literature, and many results obtained in the field are now partly summarised in the books by Libai and Simmonds (1998) and Chróścielewski et al. (2004).

In this entry the extended resultant thermomechanics of shells proposed by Pietraszkiewicz (2011) is briefly presented. The local, resultant 2D balance laws of mass, linear and angular momentum, and energy as well as the entropy inequality for shells are constructed as the *exact resultant implications* of corresponding 3D laws of rational continuum thermodynamics. As compared with the results by Simmonds (1984, 2012) the following refinements are introduced:

- The resultant laws are formulated on the shell base surface which is taken to be the material surface during the entire shell motion.
- An additional stress power, called an interstitial working, is introduced on the 2D level, which completes the initially approximate resultant 2D balance of energy expressed through the 2D stress and strain measures alone.

- The extra surface heat and entropy supplies, following from non-uniform distribution of temperature across the thickness, are accommodated by three extra surface fields.

The kinematic structure of the resultant shell theory is that of the Cosserat surface with the translation vector and rotation tensor fields as the only kinematic field variables (Pietraszkiewicz 2018). The structure of the extended resultant 2D thermomechanical laws for shells reminds somewhat that of corresponding 3D laws of extended thermodynamics, see for example Müller and Ruggeri (1998).

Basic Principles

Within 3D continuum thermodynamics one assumes that all material bodies possess mass, sustain forces and torques, convert energy, and basic laws of thermodynamics are valid for any part $\mathcal{P}$ of the body $\mathcal{B}$.

To describe the mechanical behaviour of $\mathcal{P}$ at any time $t \in T$ one assumes the following primitive quantities to be meaningful: the mass $\mathsf{M}(\mathcal{P},t)$, the mass production $\mathsf{C}(\mathcal{P},t)$, the linear momentum vector $\mathbf{L}(\mathcal{P},t)$, the total force vector $\mathbf{F}(\mathcal{P},t)$, the angular momentum vector $\mathbf{A}_{\mathrm{o}}(\mathcal{P},t)$, and the total torque vector $\mathbf{T}_{\mathrm{o}}(\mathcal{P},t)$. The latter two quantities are defined in an inertial frame $(\mathrm{o},\mathbf{e}_i)$ relative to a point o of the three-dimensional (3D) physical space $\mathcal{E}$ with V as its translation 3D vector space, and where $\mathbf{e}_i \in \mathrm{V}$, $i = 1,2,3$, are orthonormal vectors. The primitive quantities are assumed to satisfy three balance laws of continuum mechanics: balances of mass, of linear momentum and of angular momentum (Truesdell and Toupin 1960; Truesdell and Noll 1965). When written in the most general, global integral-impulse form these laws are:

$$\mathsf{M}\,|_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathsf{C}\,dt\,, \quad \mathbf{L}\,|_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathbf{F}\,dt\,, \quad \mathbf{A}_{\mathrm{o}}\,|_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathbf{T}_{\mathrm{o}}\,dt\,. \tag{1}$$

When the theory is designed to account for thermal effects, one assumes additional primitive quantities to be meaningful: the total energy $\mathsf{U}(\mathcal{P},t)$, the heating $\mathsf{Q}(\mathcal{P},t)$, the entropy $\mathsf{H}(\mathcal{P},t)$, and the entropy flux $\mathsf{J}(\mathcal{P},t)$. It is generally accepted that these quantities have to satisfy no more than two laws of continuum thermodynamics. However, while the form of energy balance is universally accepted, there is no general agreement which specific form should take The 2nd Law. One may consult reviews (Muschik et al. 2001; Muschik 2008) and books (Müller 2007; Badur 2009), where many references to historic papers and books on various formulations of continuum thermodynamics are given.

Within rational thermodynamics developed by Truesdell and Toupin 1960), Truesdell and Noll (1965) and Truesdell (1984), which is used here, the two laws of thermodynamics are the balance of energy (also called The 1st Law) and the entropy inequality (also called The 2nd Law) given by

$$\mathsf{U}\,|_{t_1}^{t_2} = \int_{t_1}^{t_2} (\mathsf{P}+\mathsf{Q})\,dt\,, \quad \mathsf{H}\,|_{t_1}^{t_2} \geq \int_{t_1}^{t_2} \mathsf{J}\,dt\,, \tag{2}$$

where $\mathsf{P}(\mathcal{P},t)$ means the mechanical power, and $\mathsf{J}(\mathcal{P},t)$ is taken in the Clausius-Duhem form, see below.

In continuum mechanics each placement $\chi(\mathcal{P},t)$ of $\mathcal{P} \in \mathcal{B}$ at time t becomes a part $\mathrm{P}(t)$ of the region $\mathrm{B}(t) = \chi(\mathcal{B},t)$ of $\mathcal{E}$. By $\mathrm{y} \in \mathrm{P(t)}$ one denotes the actual place of material particle and by $\mathbf{y} = \mathrm{y} - \mathrm{o}$ its position vector in the inertial frame $(\mathrm{o},\mathbf{e}_i)$. Then $\mathrm{P} \subset \mathrm{B}$ is the region of $\mathcal{E}$ occupied by $\mathcal{P}$ in the reference placement $\kappa(\mathcal{P})$ associated here with $t = 0$, while $\mathrm{x} \in \mathrm{B}$ is the reference place of material particle and $\mathbf{x} = \mathrm{x} - \mathrm{o}$ its position vector in the same inertial frame $(\mathrm{o},\mathbf{e}_i)$.

In the shell-like body the boundary surface $\partial\mathrm{B}$ of the reference region B consists of three separable parts: the upper M^{+} and the lower M^{-} shell faces, and the lateral boundary surface $\partial\mathrm{B}^{*}$

such that $\partial \mathrm{B} = M^{+} \cup M^{-} \cup \partial \mathrm{B}^{*}$, $M^{+} \cap M^{-} = \varnothing$. Relative to the origin $\mathrm{o} \in \mathcal{E}$ of the inertial frame the position vectors $\mathbf{x}$ and $\mathbf{y}$ are usually represented by

$$\mathbf{x}(x,\xi) = \boldsymbol{x}(x) + \xi \boldsymbol{n}(x), \quad \mathbf{y}(x,\xi,t) = \boldsymbol{y}(x,t) + \mathbf{z}(x,\xi,t), \quad \mathbf{z}(x,0,t) = \mathbf{0}. \tag{3}$$

Here $\boldsymbol{x}(x) = \mathbf{x}(x,0)$ is the position vector of corresponding point of some reference shell base surface $M \subset \mathcal{E}$, $\boldsymbol{n}(x)$ is the unit normal vector orienting M, $\xi \in [-h^{-}(x), h^{+}(x)]$ is the distance along $\boldsymbol{n}$ from M to x with $h = h^{-} + h^{+}$ the initial shell thickness, $\boldsymbol{y}(x,t)$ is the position vector of the actual shell base surface $M(t)$, and $\mathbf{z}(x,\xi,t)$ is a deviation of $\mathrm{y} \in \mathrm{B(t)}$ from $M(t)$.

Each placement $\mathrm{P}(t)$ of the moving shell-like body can be represented through a part $\Pi(t)$ of the shell base surface $M(t) \subset \mathcal{E}$ taken here to be the material surface, i.e. consisting of the same material particles during the shell motion. By $y \in \Pi(t)$ one denotes a point of $\Pi(t)$ and by $\boldsymbol{y} = y - \mathrm{o}$ its position vector in the inertial frame. Then $\Pi \subset M$ represents a part of M, while $x \in \Pi$ is the point of Π and $\boldsymbol{x} = x - \mathrm{o}$ its position vector in the same inertial frame.

Under appropriate smoothness requirements the mechanical primitive quantities can be expressed as the following volume and surface integrals of their densities, written here with respect to the reference placement:

$$\mathsf{M} = \iiint_{\mathrm{P}} \rho_R \,\mathrm{dv}, \quad \mathsf{C} = \iiint_{\mathrm{P}} \mathrm{c}_R \,\mathrm{dv}, \quad \mathbf{L} = \iiint_{\mathrm{P}} \rho_R \dot{\mathbf{y}} \,\mathrm{dv}, \quad \mathbf{A}_{\mathrm{o}} = \iiint_{\mathrm{P}} \mathbf{y} \times \rho_R \dot{\mathbf{y}} \mathrm{dv}, \tag{4}$$

$$\mathbf{F} = \iiint_{\mathrm{P}} \rho_R \mathbf{b} \,\mathrm{dv} + \iint_{\partial \mathrm{P}} \mathbf{t}_{\mathrm{n}} \mathrm{da}, \quad \mathbf{T}_{\mathrm{o}} = \iiint_{\mathrm{P}} \mathbf{y} \times \rho_R \mathbf{b} \,\mathrm{dv} + \iint_{\partial \mathrm{P}} \mathbf{y} \times \mathbf{t}_{\mathrm{n}} \mathrm{da}. \tag{5}$$

Here $\rho_R(\mathrm{x},t) > 0$ and $\mathrm{c}_R(\mathrm{x},t)$ are the referential mass and mass production (densities) per unit volume of B, $\mathbf{b}(\mathrm{x},t)$ is the body force (density) per unit mass of B, $\dot{\mathbf{y}}(\mathrm{x},t)$ is the 3D velocity field, and $\mathbf{t}_{\mathrm{n}}(\mathrm{x},t)$ is the contact force (density) per unit area of $\partial \mathrm{P}$ with the unit normal vector $\mathbf{n}(\mathrm{x},t)$ orienting $\partial \mathrm{P}$.

One can define the following resultant 2D surface fields:

$$\rho = \int_{-}^{+} \rho_R \mu \,\mathrm{d}\xi, \quad c = \int_{-}^{+} \mathrm{c}_R \mu \,\mathrm{d}\xi, \quad \boldsymbol{l} = \int_{-}^{+} \rho_R \dot{\mathbf{y}} \mu \,\mathrm{d}\xi, \quad \boldsymbol{k} = \int_{-}^{+} \mathbf{y} \times \rho_R \dot{\mathbf{y}} \mu \,\mathrm{d}\xi, \tag{6}$$

$$\boldsymbol{n}_{\nu} = \int_{-}^{+} \mathbf{t}_{\mathrm{n}} \mu \,\mathrm{d}\xi, \quad \boldsymbol{m}_{\nu} = \int_{-}^{+} \mathbf{z} \times \mathbf{t}_{\mathrm{n}} \mu \,\mathrm{d}\xi, \quad \int_{-}^{+} \equiv \int_{-h^{-}}^{h^{+}}, \tag{7}$$

$$\rho \boldsymbol{f} = \int_{-}^{+} \rho_R \mathbf{b} \mu \,\mathrm{d}\xi + \left(\mathbf{t}_{\mathrm{n}} \alpha \right)\big|_{-}^{+}, \quad \rho \boldsymbol{c} = \int_{-}^{+} \mathbf{z} \times \rho_R \mathbf{b} \mu \,\mathrm{d}\xi + \left(\mathbf{z} \times \mathbf{t}_{\mathrm{n}} \alpha \right)\big|_{-}^{+}, \tag{8}$$

where $(\mathbf{x})\big|_{-}^{+} \equiv \mathbf{x}^{+} - \mathbf{x}^{-}$ and the geometric parameters $\mu, \alpha^{\pm}$ are given in (Konopińska and Pietraszkiewicz 2007, A.15 – A.17).

In (6) - (8), $\rho(x,t) > 0$ and $c(x,t)$ are the referential surface mass and mass production (densities), $\boldsymbol{l}(x,t)$ and $\boldsymbol{k}(x,t)$ are the surface linear momentum and angular momentum vectors per unit area of M, while $\boldsymbol{f}(x,t)$ and $\boldsymbol{c}(x,t)$ are the surface force and couple vectors per unit mass of M, respectively. Additionally, $\boldsymbol{n}_{\nu}(x,t)$ and $\boldsymbol{m}_{\nu}(x,t)$ are the surface contact stress and couple-stress vectors describing internal mechanical interactions between the shell parts at the internal boundary $\partial \Pi \setminus \partial M_f$.

With the help of (6) - (8) the mechanical primitive quantities can also be expressed through their resultant 2D representatives,

$$\mathsf{M} = \iint_{\Pi} \rho \, da, \quad \mathsf{C} = \iint_{\Pi} c \, da, \tag{9}$$

$$\mathsf{L} = \iint_{\Pi} \boldsymbol{l}\, da\,, \quad \mathsf{F} = \iint_{\Pi} \rho \boldsymbol{f}\, da + \int_{\partial\Pi\backslash\partial M_f} \boldsymbol{n}_\nu\, ds + \int_{\partial\Pi\cap\partial M_f} \boldsymbol{n}^*\, ds\,, \tag{10}$$

$$\begin{aligned} \mathsf{A}_o = \int_{\Pi} (\boldsymbol{k} + \boldsymbol{y}\times\boldsymbol{l})\, da\,, \quad \mathsf{T}_o = \int_{\Pi} (\rho\boldsymbol{c} + \boldsymbol{y}\times\rho\boldsymbol{f})\, da + \int_{\partial\Pi\backslash\partial M_f} (\boldsymbol{m}_\nu + \boldsymbol{y}\times\boldsymbol{n}_\nu)\, ds \\ + \int_{\partial\Pi\cap\partial M_f} (\boldsymbol{m}^* + \boldsymbol{y}\times\boldsymbol{n}^*)\, ds\,, \end{aligned} \tag{11}$$

where $\boldsymbol{n}^*, \boldsymbol{m}^*$ are just the external resultant boundary force and couple vectors assigned along a part $\partial M_f \subset \partial M$, which are statically equivalent to distribution of external tractions $\mathbf{t}^*$ applied on $\partial \mathrm{B}_f^*$.

Similarly, the primitive quantities associated with The 1st and 2nd Laws can be expressed with respect to the reference placement by the following integrals:

$$\mathsf{U} = \iiint_{\mathrm{P}} \rho_R \mathrm{u}\, \mathrm{dv}\,, \quad \mathsf{P} = \iiint_{\mathrm{P}} \mathrm{p}\, \mathrm{dv} + \iint_{\partial\mathrm{P}} \mathrm{p_n}\, \mathrm{da}\,, \quad \mathsf{H} = \iiint_{\mathrm{P}} \rho_R \eta\, \mathrm{dv}\,, \tag{12}$$

$$\mathsf{Q} = \iiint_{\mathrm{P}} \rho_R \mathrm{r}\, \mathrm{dv} - \iint_{\partial\mathrm{P}} \mathrm{p_n}\, \mathrm{da}\,, \quad \mathsf{J} = \iiint_{\mathrm{P}} \rho_R \mathrm{k}\, \mathrm{dv} - \iint_{\partial\mathrm{P}} \mathrm{j_n}\, \mathrm{da}\,. \tag{13}$$

Here $\mathrm{u}(\mathrm{x},t)$, $\eta(\mathrm{x},t)$, $\mathrm{r}(\mathrm{x},t)$, and $\mathrm{k}(\mathrm{x},t)$ are the 3D (referential) total energy, entropy, heat supply, and entropy supply (densities), all per unit mass of B, $\mathrm{p}(\mathrm{x},t)$ is the 3D mechanical power per unit volume of B, while $\mathrm{p_n}(\mathrm{x},t)$, $\mathrm{q_n}(\mathrm{x},t)$, and $\mathrm{j_n}(\mathrm{x},t)$ are the 3D contact power, heat, and entropy fluxes through the boundary $\partial\mathrm{P}$, respectively.

One can again define the resultant surface fields:

$$\rho u = \int_-^+ \rho_R \mathrm{u} \mu\, \mathrm{d}\xi\,, \quad p = \int_-^+ \mathrm{p}\mu\, \mathrm{d}\xi\,, \quad p_\nu = \int_-^+ \mathrm{p_n}\mu\, \mathrm{d}\xi\,, \quad \rho\eta = \int_-^+ \rho_R \eta\mu\, \mathrm{d}\xi\,, \tag{14}$$

$$\rho r = \int_-^+ \rho_R \mathrm{r}\mu\, \mathrm{d}\xi - \left(\mathrm{q_n}\alpha\right)\Big|_-^+\,, \quad q_\nu = \int_-^+ \mathrm{q_n}\mu\, \mathrm{d}\xi\,, \tag{15}$$

$$\rho k = \int_-^+ \rho_R \mathrm{k}\mu\, \mathrm{d}\xi - \left(\mathrm{j_n}\alpha\right)\Big|_-^+\,, \quad j_\nu = \int_-^+ \mathrm{j_n}\mu\, \mathrm{d}\xi\,. \tag{16}$$

In (14) - (16), $u(x,t)$, $\eta(x,t)$, $r(x,t)$, and $k(x,t)$ are the resultant total energy, entropy, heat supply, and entropy supply (densities), all per unit mass of M, $p(x,t)$ is the resultant mechanical power per unit area of M, while $p_\nu(x,t)$, $q_\nu(x,t)$, and $j_\nu(x,t)$ are the resultant contact mechanical power, heat, and entropy fluxes through the internal boundary $\partial\Pi$, respectively.

With the help of (14) - (16) the quantities (12) and (13) can also be expressed through their 2D representatives,

$$\mathsf{U} = \iint_{\Pi} \rho u\, da\,, \quad \mathsf{P} = \iint_{\Pi} p\, da + \int_{\partial\Pi\backslash\partial M_f} p_\nu\, ds + \int_{\partial\Pi\cap\partial M_f} p^* ds\,, \quad \mathsf{H} = \iint_{\Pi} \rho\eta\, da\,, \tag{17}$$

$$\mathsf{Q} = \iint_{\Pi} \rho r\, da - \int_{\partial\Pi\backslash\partial M_h} q_\nu\, ds - \int_{\partial\Pi\cap\partial M_h} q^* ds\,, \tag{18}$$

$$\mathsf{J} = \iint_{\Pi} \rho k\, da - \int_{\partial\Pi\backslash\partial M_h} j_\nu\, ds - \int_{\partial\Pi\cap\partial M_h} j^* ds\,, \tag{19}$$

where p^* is the external resultant boundary power flux assigned along ∂M_f, while q^* and j^* are the external resultant boundary heat and entropy fluxes given along a part $\partial M_h \subset \partial M$, which are thermally equivalent to distributions of 3D heat q^* and entropy j^* fluxes assigned on $\partial \mathrm{B}_h^* \subset \partial \mathrm{B}^*$.

By the Cauchy postulate extended to the 2D thermal fields, the contact surface quantities $\boldsymbol{n}_\nu$, $\boldsymbol{m}_\nu$, q_ν, and j_ν can be represented through the respective surface stress resultant $\boldsymbol{N}(x,t) \in V \otimes T_x M$ and stress couple $\boldsymbol{M}(x,t) \in V \otimes T_x M$ tensors of the 1st Piola-Kirchhoff type, as well as the respective referential heat $\boldsymbol{q}(x,t) \in T_x M$ and entropy $\boldsymbol{j}(x,t) \in T_x M$ flux vectors according to

$$\boldsymbol{n}_\nu = \boldsymbol{N}\boldsymbol{\nu}, \quad \boldsymbol{m}_\nu = \boldsymbol{M}\boldsymbol{\nu}, \quad p_\nu = \boldsymbol{p}\cdot\boldsymbol{\nu}, \quad q_\nu = \boldsymbol{q}\cdot\boldsymbol{\nu}, \quad j_\nu = \boldsymbol{j}\cdot\boldsymbol{\nu}. \tag{20}$$

In these relations $\boldsymbol{\nu}\in T_x M$ is the unit vector externally normal to $\partial\Pi$, and $T_x M$ is the 2D vector space tangent to M at $x \in M$.

In what follows one assumes, as is usual in solid mechanics, that mass is not produced during the process, $\mathsf{C}\equiv 0$. Hence, the balance of mass $(1)_1$ is identically satisfied.

If time derivatives of the set functions $\mathsf{L}(\mathcal{P},t)$, $\mathsf{A}_\mathrm{o}(\mathcal{P},t)$, $\mathsf{U}(\mathcal{P},t)$, $\mathsf{P}(\mathcal{P},t)$, and $\mathsf{H}(\mathcal{P},t)$ exist for all $t\in T$ one can write

$$\mathsf{L}|_{t_1}^{t_2} = \int_{t_1}^{t_2}\dot{\mathsf{L}}\,dt, \quad \mathsf{A}_\mathrm{o}|_{t_1}^{t_2} = \int_{t_1}^{t_2}\dot{\mathsf{A}}_\mathrm{o}\,dt, \quad \mathsf{U}|_{t_1}^{t_2} = \int_{t_1}^{t_2}\dot{\mathsf{U}}\,dt, \quad \mathsf{P}|_{t_1}^{t_2} = \int_{t_1}^{t_2}\dot{\mathsf{P}}\,dt, \quad \mathsf{H}|_{t_1}^{t_2} = \int_{t_1}^{t_2}\dot{\mathsf{H}}\,dt. \tag{21}$$

Then using the 2D representations (9) - (19), one obtains

$$\frac{d}{dt}\iint_\Pi \boldsymbol{l}\,da = \iint_\Pi \dot{\boldsymbol{l}}\,da, \quad \frac{d}{dt}\iint_\Pi (\boldsymbol{k}+\boldsymbol{y}\times\boldsymbol{l})\,da = \iint_\Pi (\dot{\boldsymbol{k}}+\dot{\boldsymbol{y}}\times\boldsymbol{l}+\boldsymbol{y}\times\dot{\boldsymbol{l}})\,da, \tag{22}$$

$$\frac{d}{dt}\iint_\Pi \rho u\,da = \iint_\Pi \rho\dot{u}\,da, \quad \frac{d}{dt}\iint_\Pi p\,da = \iint_\Pi \dot{p}\,da, \quad \frac{d}{dt}\iint_\Pi \rho\eta\,da = \iint_\Pi \rho\dot{\eta}\,da, \tag{23}$$

and the four remaining laws of mechanics and thermodynamics for the shell-like body become

$$\iint_\Pi (\rho\boldsymbol{f}-\dot{\boldsymbol{l}})\,da + \int_{\partial\Pi\backslash\partial M_f}\boldsymbol{n}_\nu\,ds + \int_{\partial\Pi\backslash\partial M_f}\boldsymbol{n}^*\,ds = \boldsymbol{0}, \tag{24}$$

$$\begin{aligned}&\iint_\Pi \{\rho\boldsymbol{c}-(\dot{\boldsymbol{k}}+\dot{\boldsymbol{y}}\times\boldsymbol{l})+\boldsymbol{y}\times(\rho\boldsymbol{f}-\dot{\boldsymbol{l}})\}\,da\\ &\quad + \int_{\partial\Pi\backslash\partial M_f}(\boldsymbol{m}_\nu+\boldsymbol{y}\times\boldsymbol{n}_\nu)\,ds + \int_{\partial\Pi\cap\partial M_f}(\boldsymbol{m}^*+\boldsymbol{y}\times\boldsymbol{n}^*)\,ds = \boldsymbol{0},\end{aligned} \tag{25}$$

$$\begin{aligned}&\iint_\Pi (\rho\dot{u}-p)\,da - \int_{\partial\Pi\backslash\partial M_f} p_\nu\,ds - \int_{\partial\Pi\cap\partial M_h} p^*ds\\ &\quad -\iint_\Pi \rho r\,da + \int_{\partial\Pi\backslash\partial M_h} q_\nu\,ds + \int_{\partial\Pi\cap\partial M_h} q^*ds = 0,\end{aligned} \tag{26}$$

$$\iint_\Pi \rho\dot{\eta}\,da - \iint_\Pi \rho k\,da + \int_{\partial\Pi\backslash\partial M_h} j_\nu\,ds + \int_{\partial\Pi\cap\partial M_h} j^*ds \geq 0. \tag{27}$$

In what follows one assumes that M be a regular geometric surface, so that any kinks, branchings and self-intersections are excluded. One also assumes that all surface fields discussed here are smooth in Π.

To (24) - (27) with (20) one can apply the surface divergence theorems:

$$\int_{\partial\Pi}\boldsymbol{a}\cdot\boldsymbol{\nu}\,ds = \iint_\Pi Div\,\boldsymbol{a}\,da, \quad \int_{\partial\Pi}\boldsymbol{S}\boldsymbol{\nu}\,ds = \iint_\Pi Div\,\boldsymbol{S}\,da, \tag{28}$$

$$\int_{\partial\Pi}\boldsymbol{a}\times\boldsymbol{S}\boldsymbol{\nu}\,ds = \iint_\Pi \{\boldsymbol{a}\times Div\,\boldsymbol{S} + \mathrm{ax}[\boldsymbol{S}(Grad\,\boldsymbol{a})^T - (Grad\,\boldsymbol{a})\boldsymbol{S}^T]\}\,da, \tag{29}$$

valid for any $\boldsymbol{a}(x,t)\in T_x M$ and $\boldsymbol{S}(x,t)\in V\otimes T_x M$, where the surface gradient and divergence operators with respect to $x\in M$ are defined as in Gurtin and Murdoch (1975), and $(\mathrm{ax}\,\mathbf{T})\in V$ means the axial vector of the skew tensor $\mathbf{T}\in V\otimes V$, $\mathbf{T}^T = -\mathbf{T}$, so that $\mathbf{T} = (\mathrm{ax}\,\mathbf{T})\times\mathbf{1}$, where $\mathbf{1}\in V\otimes V$ is the 3D identity tensor. Then, after some transformations one obtains the following four local laws of resultant shell thermomechanics in the Lagrangian description valid in any $\Pi\in M$:

$$Div\,\boldsymbol{N}+\rho\boldsymbol{f} = \dot{\boldsymbol{l}}, \quad Div\,\boldsymbol{M}+\mathrm{ax}(\boldsymbol{N}\boldsymbol{F}^T-\boldsymbol{F}\boldsymbol{N}^T)+\rho\boldsymbol{c} = \dot{\boldsymbol{k}}+\dot{\boldsymbol{y}}\times\boldsymbol{l}, \tag{30}$$

$$\rho\dot{u} - (p + Div\,\boldsymbol{p}) - (\rho r - Div\,\boldsymbol{q}) = 0, \tag{31}$$

$$\rho\dot{\eta} - (\rho k - Div\,\boldsymbol{j}) \geq 0, \tag{32}$$

where $\boldsymbol{F} = Grad\,\boldsymbol{y} \in V \otimes T_x M$ is the surface deformation gradient.

The corresponding dynamic and thermal boundary conditions are

$$\boldsymbol{n}^* - \boldsymbol{N}\boldsymbol{\nu} = \boldsymbol{0}\,, \quad \boldsymbol{m}^* - \boldsymbol{M}\boldsymbol{\nu} = \boldsymbol{0}\,, \quad p^* - \boldsymbol{p}\cdot\boldsymbol{\nu} = 0 \quad \text{along } \partial M_f\,, \tag{33}$$

$$q^* - \boldsymbol{q}\cdot\boldsymbol{\nu} = 0\,, \quad j^* - \boldsymbol{j}\cdot\boldsymbol{\nu} = 0 \quad \text{along } \partial M_h\,. \tag{34}$$

The relations (30) - (34) are formally *exact implications* of the global laws of continuum thermodynamics (1), (2), with (21) and 2D representations (9) - (11), (17) - (19), for the shell-like body represented during motion by the material base surface $M(t)$, which in the reference placement is M.

Modified Resultant Energy Balance

It was noted in Pietraszkiewicz (2011) that during the through-the-thickness integration some part of the 3D mechanical power following from the Piola stress tensor $\mathbf{P}$ acting on surfaces in B parallel to M as well as from self-equilibrated distributions across the shell cross section of $\mathbf{P}$, body forces $\mathbf{b}$ and boundary tractions $\mathbf{t}^*$ is not accounted for. In fact, Pietraszkiewicz et al. (2006) proved explicitly that the 3D stress power can be expressed through the resultant 2D stress power *plus* an additional stress power not expressible through $\boldsymbol{N}, \boldsymbol{M}$. As a result, one can write the resultant 2D balance of mechanical energy symbolically as $\mathsf{P}_e - \mathsf{S}_e = \mathsf{K}_e$, where indices e mean that these quantities are *effective* quantities calculated using only the surface fields defined on the material base surface. In particular, if S and S_e are given through their 2D representatives then

$$\mathsf{S} = \iint_{\Pi} \sigma\, da\,, \quad \mathsf{S}_e = \iint_{\Pi} \sigma_e\, da\,, \quad \sigma_e < \sigma\,. \tag{35}$$

In continuum mechanics, the total energy $\mathsf{U}(\mathcal{P},t)$ is often decomposed into the kinetic energy $\mathsf{K}(\mathcal{P},t)$ and the internal energy $\mathsf{E}(\mathcal{P},t)$,

$$\mathsf{U} = \mathsf{K} + \mathsf{E}\,, \quad \mathsf{E} = \iiint_{\mathrm{P}} \rho_R \varepsilon\, \mathrm{dv} = \iint_{\Pi} \rho\varepsilon\, da\,, \quad \rho\varepsilon = \int_-^+ \rho_R \varepsilon \mu\, \mathrm{d}\xi\,. \tag{36}$$

On the other hand, the mechanical power $\mathsf{P}(\mathcal{P},t)$ can be related to K by $\mathsf{P} = \dot{\mathsf{K}} + \mathsf{S}$. Then the balance of energy $(2)_1$ can be stated in the alternative simpler form

$$\dot{\mathsf{E}} = \mathsf{S} + \mathsf{Q}\,, \quad \text{or} \quad \mathsf{E}\,\big|_{t_1}^{t_2} = \int_{t_1}^{t_2} (\mathsf{S} + \mathsf{Q})\, dt\,. \tag{37}$$

From (37) follows the simpler form of local, resultant balance of energy

$$\rho\dot{\varepsilon} - \sigma - (\rho r - Div\,\boldsymbol{q}) = 0. \tag{38}$$

Thanks to Libai and Simmonds (1983, 1998), Chróścielewski et al. (2004), Pietraszkiewicz et al. (2006), the integrand of S_e can also be given in the following coordinate-free form:

$$\sigma_e = \boldsymbol{N} \bullet \boldsymbol{E}^{\mathrm{o}} + \boldsymbol{M} \bullet \boldsymbol{K}^{\mathrm{o}}\,, \tag{39}$$

$$\boldsymbol{E} = \boldsymbol{J}\mathbf{F} - \boldsymbol{Q}\boldsymbol{I}\,, \quad \boldsymbol{K} = \boldsymbol{C}\mathbf{F} - \boldsymbol{Q}\boldsymbol{B}\,, \tag{40}$$

$$\boldsymbol{E}^{\mathrm{o}} = \boldsymbol{Q}\frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{E}) = Grad\,\boldsymbol{\upsilon} - \boldsymbol{\Omega}\boldsymbol{F}\,, \quad \boldsymbol{K}^{\mathrm{o}} = \boldsymbol{Q}\frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{K}) = Grad\,\boldsymbol{\omega}\,. \tag{41}$$

In definitions (40) of the natural surface stretch $\boldsymbol{E}(x,t)$ and bending $\boldsymbol{K}(x,t)$ tensors, $\boldsymbol{I} \in V \otimes T_x M$ and $\boldsymbol{J} \in V \otimes T_y M(t)$ are the inclusion operators at $x \in M$ and $y \in M(t)$, see Gurtin and Murdoch (1975), $\boldsymbol{B} \in V \otimes T_x M$ and $\boldsymbol{C} \in V \otimes T_y M(t)$ are the structure tensors of the shell in the reference and actual placement, respectively, and $\mathbf{F} \in T_y M(t) \otimes T_x M$ is the tangential surface deformation gradient such that $dy = \mathbf{F} dx$, $\boldsymbol{F} = \boldsymbol{J}\mathbf{F}$. The co-rotational time derivative $(.)^{o}$ is defined in (41) through the rotation tensor $\boldsymbol{Q} = \boldsymbol{d}_i \otimes \boldsymbol{t}_i$, $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$, $\det \boldsymbol{Q} = +1$, where $\boldsymbol{d}_i(x,t)$ and $\boldsymbol{t}_i(x)$, $i = 1,2,3$, are the orthonormal base vectors (directors) in the actual and reference placement, respectively. Moreover, now

$$\boldsymbol{u} = \boldsymbol{y} - \boldsymbol{x}, \quad \boldsymbol{\upsilon} = \dot{\boldsymbol{y}} = \dot{\boldsymbol{u}}, \quad \boldsymbol{\omega} = \mathrm{ax}(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T), \quad \boldsymbol{\Omega} = \boldsymbol{\omega} \times \mathbf{1}, \tag{42}$$

where $\boldsymbol{u}(x,t)$ is the surface translation vector and $\boldsymbol{Q}(x,t)$ is the surface rotation tensor. The fields $\boldsymbol{y}$ (or $\boldsymbol{u}$) and $\boldsymbol{Q}$ are independent kinematic variables of the shell motion. Thus, the complementary to (33) displacement boundary conditions are

$$\boldsymbol{y}^* - \boldsymbol{y} = \mathbf{0}, \quad \boldsymbol{Q}^* - \boldsymbol{Q} = \mathbf{0} \quad \text{along } \partial M_d = \partial M \setminus \partial M_f . \tag{43}$$

In the resultant balance of energy (38) the resultant stress power σ is required, while only its effective part σ_e is available in (39). Use of σ_e in place of σ in (38) as in Simmonds (1984, 2012) introduces indefinable error into the resultant energy balance (38). To compensate this error, one can introduce an additional stress power $\mathsf{W}(\mathcal{P},t)$ of the shell-like body, called here *the interstitial working* after Dunn and Serrin (1985), such that $\mathsf{S} = \mathsf{S}_e + \mathsf{W}$. For any $\Pi \subset M$ the interstitial working may be represented locally as

$$\mathsf{W} = \int_{\partial\Pi} w_\nu \, ds = \iint_\Pi Div\, \boldsymbol{w}\, da , \tag{44}$$

where $w_\nu(x,t)$ is the surface contact interstitial working (density) and $\boldsymbol{w}(x,t) \in T_x M$ is the corresponding surface interstitial working flux vector such that $w_\nu = \boldsymbol{w} \cdot \boldsymbol{\nu}$, so that now $\sigma = \sigma_e + Div\boldsymbol{w}$. Then the local, resultant balance of energy (38) is modified into

$$\rho\dot{\varepsilon} - (\boldsymbol{N} \bullet \boldsymbol{E}^{o} + \boldsymbol{M} \bullet \boldsymbol{K}^{o} + Div\boldsymbol{w}) - (\rho r - Div\boldsymbol{q}) = 0 . \tag{45}$$

The resultant equation (45) can now be regarded as an *exact implication* of the global 3D balance of energy (37).

Modified Resultant Entropy Inequality

The local resultant entropy inequality in the form (32) is entirely decoupled from other local resultant balance laws (30) and (45).

In continuum thermodynamics coupling of The 2nd Law $(2)_2$ with other balance laws $(1)_{2,3}$ and with $(2)_1$ is achieved by introducing the absolute 3D temperature field $\theta(\mathrm{x},t) > 0$, through which the fields $\mathrm{k}(\mathrm{x},t)$ and $\mathrm{j_n}(\mathrm{x},t)$ in $(13)_2$ are related to those $\mathrm{r}(\mathrm{x},t)$ and $\mathrm{q_n}(\mathrm{x},t)$ in $(13)_1$. In rational continuum thermomechanics these relations are taken as $\mathrm{k} = \mathrm{r}/\theta$ and $\mathrm{j_n} = \mathbf{q} \cdot \mathbf{n} / \theta$. The 3D entropy inequality in the form

$$\iiint_{\mathrm{P}} \rho_{\mathrm{R}} \dot{\eta}\, \mathrm{dv} \geq \iiint_{\mathrm{P}} \rho_R \frac{\mathrm{r}}{\theta} \mathrm{dv} - \iint_{\partial \mathrm{P}} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} \mathrm{da} \tag{46}$$

is usually called the Clausius-Duhem inequality, (see Truesdell and Toupin 1960; Truesdell 1984).

Three different 2D temperature fields appear naturally in shell thermodynamics: a reference temperature associated with the base surface M, and two temperatures of the upper and lower shell faces M^+ and M^-. Postulating some reasonable relations between the three surface temperatures one can reduce the number of independent 2D temperature fields to two or to one, whichever is appropriate. In particular, Murdoch (1976a) proposed to use only one common temperature field associated with M, and this approach has recently been used by Eremeyev and Pietraszkiewicz

(2009). Temperatures of the upper and lower shell faces as independent fields were used by Zhilin (1976), and Eremeyev and Zubov (2008). Naghdi (1972), and Green and Naghdi (1979) used the thickness-averaged temperature and its derivative in the transverse normal direction evaluated on M as independent fields, while Simmonds(2012) used the maximal and minimal temperatures across the thickness and introduced their average and difference temperatures as independent variables. Recently Eremeyev and Pietraszkiewicz (2011) developed the resultant, thermomechanic, quasistatic model of phase transitions in shells, where the referential mean temperature and its deviation suggested by Murdoch (1976b) were used. Any such proposal leads to a slightly different structure of the thermodynamic initial-boundary value problem for shells. In particular, for two independent 2D temperature fields one needs two independent 2D energy balance equations. Since the shell thermodynamic theories mentioned above are not entirely resultant ones, they introduce an indefinable error into the 2D energy balance and entropy inequality.

In this entry the surface mean referential temperature $\theta(x,t)$ is defined by

$$\frac{1}{\theta}=\frac{1}{2}\left(\frac{1}{\theta_+}+\frac{1}{\theta_-}\right),\quad \frac{1}{\theta_+}=\frac{1}{\theta}-\frac{1}{2}\left(\frac{1}{\theta_-}-\frac{1}{\theta_+}\right),\quad \frac{1}{\theta_-}=\frac{1}{\theta}+\frac{1}{2}\left(\frac{1}{\theta_-}-\frac{1}{\theta_+}\right), \tag{47}$$

where θ_+ and θ_- are values of temperature on the upper and lower shell faces M^+ and M^-, respectively. The use of so defined θ itself does not introduce any approximation. Then the through-the-thickness integration in (46) with (47) allows one to represent the Clausius-Duhem inequality in the resultant form

$$\begin{aligned}\iint_{\Pi}\left\{\rho\dot{\eta}-\rho\left(\frac{r}{\theta}+s\right)+\frac{1}{\theta}Div\boldsymbol{q}-\frac{1}{\theta^2}\boldsymbol{q}\cdot\boldsymbol{g}+Div\,\boldsymbol{s}\right\}da \\ +\int_{\partial\Pi\cap\partial M_h}\left\{\frac{q^*}{\theta^*}+s^*-\left(\frac{q_\nu}{\theta}+s_\nu\right)\right\}ds\geq 0\,,\end{aligned} \tag{48}$$

where

$$\boldsymbol{g}=Grad\,\theta\in T_xM\,,\quad \rho r=\int_-^+\rho_R r\mu\,d\xi-(\mathbf{q}\cdot\mathbf{n}\alpha)|_-^+\,, \tag{49}$$

$$\rho s=\int_-^+\left(\frac{1}{\theta}-\frac{1}{\theta}\right)\rho_R r\mu\,d\xi+\frac{1}{2}\left(\frac{1}{\theta_-}-\frac{1}{\theta_+}\right)(\mathbf{q}\cdot\mathbf{n}\alpha)|_-^+\,, \tag{50}$$

$$q_\nu=\boldsymbol{q}\cdot\boldsymbol{\nu}=\int_-^+\mathbf{q}\cdot\mathbf{n}^*\mu\,d\xi\,,\quad s_\nu=\boldsymbol{s}\cdot\boldsymbol{\nu}=\int_-^+\left(\frac{1}{\theta}-\frac{1}{\theta}\right)\mathbf{q}\cdot\mathbf{n}^*\mu\,d\xi\,, \tag{51}$$

$$q^*=\boldsymbol{q}^*\cdot\boldsymbol{\nu}=\int_-^+\mathbf{q}^*\cdot\mathbf{n}^*\mu\,d\xi\,,\quad s^*=\boldsymbol{s}^*\cdot\boldsymbol{\nu}=\int_-^+\left(\frac{1}{\theta^*}-\frac{1}{\theta^*}\right)\mathbf{q}^*\cdot\mathbf{n}^*\mu\,d\xi\,, \tag{52}$$

and the geometric parameters $\mu,\alpha^{\pm},\mathbf{n}^{\pm},\mathbf{n}^*$ are given by Konopińska and Pietraszkiewicz (2007, A.15 – A.17).

With definitions (49) - (52), the relations between the resultant fields appearing in (31), (32) and (34) become

$$k=\frac{r}{\theta}+s\,,\quad \boldsymbol{j}=\frac{\boldsymbol{q}}{\theta}+\boldsymbol{s}\,,\quad \boldsymbol{j}^*=\frac{\boldsymbol{q}^*}{\theta}+\boldsymbol{s}^*. \tag{53}$$

The extra surface fields $s,\boldsymbol{s},\boldsymbol{s}^*$ in (53) take into account the extra surface heat and entropy supplies following from non-uniform distribution across the shell thickness of the temperature field θ, which now enters (48) only through its value θ on base surface M. Presence of the extra fields in (48) assures that the resultant form of Clausius–Duhem inequality (48) still remains an exact implication of the 3D principle (46).

With usual continuity assumptions the local form of (48) is

$$\rho\dot{\eta}-\frac{1}{\theta}(\rho r-Div\boldsymbol{q})-\rho s+Div\boldsymbol{s}-\frac{1}{\theta^2}\boldsymbol{q}\cdot\boldsymbol{g}\geq 0 \quad \text{in } \Pi\subset M\,, \tag{54}$$

$$\frac{q^*}{\theta^*}+s^*-\left(\frac{\boldsymbol{q}}{\theta}+\boldsymbol{s}\right)\cdot\boldsymbol{\nu}\geq 0 \quad \text{along } \partial M_h\,. \tag{55}$$

One can solve the exact, resultant balance of energy (45) for $\rho r-Div\boldsymbol{q}$ and use the result in (54), which gives

$$\theta\rho\dot{\eta}-\rho\dot{\varepsilon}+(\boldsymbol{N}\bullet\boldsymbol{E}^{\mathrm{o}}+\boldsymbol{M}\bullet\boldsymbol{K}^{\mathrm{o}}+Div\boldsymbol{w})-\theta\rho s-\frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g}+\theta\,Div\boldsymbol{s}\geq 0 \quad \text{in } \Pi\subset M\,. \tag{56}$$

Upon introducing the surface free energy (density) $\psi(x,t)$ by $\psi=\varepsilon-\theta\eta$, one has $\theta\dot{\eta}-\dot{\varepsilon}=-\dot{\psi}-\dot{\theta}\eta$, and (56) takes the final form

$$-\rho\dot{\psi}-\dot{\theta}\rho\eta+\boldsymbol{N}\bullet\boldsymbol{E}^{\mathrm{o}}+\boldsymbol{M}\bullet\boldsymbol{K}^{\mathrm{o}}+Div\boldsymbol{w}-\theta\rho s-\frac{1}{\theta}\boldsymbol{q}\cdot\boldsymbol{g}+\theta\,Div\boldsymbol{s}\geq 0 \quad \text{in } \Pi\subset M\,. \tag{57}$$

The local resultant 2D entropy inequality (57) can now be regarded as an *exact implication* of the global Clausius–Duhem inequality (46) as well.

Remarks on Constitutive Equations

The local resultant 2D balance laws (30), (45) and the inequality (57) are expressed through 16 fields, which together form the *shell thermomechanic process* over the domain $M\times T$. Different groups of the fields play different roles in the process. The fields $\boldsymbol{y},\boldsymbol{Q},\theta$ constitute the basic thermo-kinematic independent field variables of the initial-boundary value problem of shell thermomechanics. That only seven scalar fields can be taken as independent field variables here follows from the fact that there are only seven scalar resultant field equations (30) and (45) to determine them. The fields $\boldsymbol{N},\boldsymbol{M},\boldsymbol{q},\varepsilon,\eta,\boldsymbol{w},s,\boldsymbol{s}$ have to be specified by appropriate material constitutive equations and the fields $\boldsymbol{l},\boldsymbol{k}$ by appropriate kinetic constitutive equations. When all the fields above are settled, the fields $\boldsymbol{f},\boldsymbol{c},r$ are supposed to be adjusted so as to satisfy the 2D balance equations (30) and (45). Every such process is called an *admissible thermomechanic process*; it is completely determined by the evolution of deformation and temperature of the shell base surface.

In the resultant shell thermomechanics specific forms of the constitutive equations can be established by two main approaches. The *direct* approach consists in developing, for a restricted class of shell-like bodies, a general structure of 2D constitutive equations satisfying some reasonable physical and mathematical requirements. Then one has to devise a suitable sets of experiments from which the appropriate material constants or functions entering the constitutive equations can be established. In the *derived* or *deductive* approach one has to devise suitable mathematical methods allowing one to deduce the 2D constitutive equations for shells as an exact, asymptotic of otherwise rational consequence of a given set of corresponding 3D constitutive equations of the parent theory.

Due to the limited space of this entry, the interested reader should consult discussion given in Pietraszkiewicz (2011) on constitutive equations of the refined resultant 2D thermomechanics of shells. There one can find some general requirements which the shell material constitutive equations must obey. Several admissible forms of the response functionals, in which also the possibility of longer-range spatial interactions is accounted for, have been proposed for constitutive equations of viscous shells with heat conduction and of thermoelastic shells. The procedure of Coleman and Noll (1963) has been used to analyse restrictions imposed by our refined entropy inequality (57) on the 2D forms of constitutive equations. Finally, several novel forms of the 2D kinetic constitutive equations obtained with the help of heuristic arguments have been provided.

Cross References

Surface Geometry, Elements
Elastic Shells, Resultant Non-linear theory
Junctions in Irregular Shell Structures

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