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Surface Geometry, Elements

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Synonyms

Differential geometry; Geometry of curves and surfaces

Definition

By geometry of a surface one usually means characterisation of its metric and curvature properties in surface curvilinear coordinates. Due to a large variety of surface shapes, it is convenient to use the common tensor notation. In shell theory, the most useful concepts are the surface covariant differentiation, description of surface curves and surface divergence theorems of vector and tensor fields.

Introduction

Geometry of a surface embedded into the three-dimensional Euclidean point space was presented in many classical monographs, for example by Eisenhart (1947) and do Carmo (1976). Within the needs of theoretical description required in shell structures, appropriate introductions were worked out as parts of the books by Green and Zerna (1954), Naghdi (1963), Chernykh (1964), Flügge (1972), Pietraszkiewicz (1977), Başar and Krätzig (2001), Ciarlet (2005), and Lebedev et al. (2010).

Here an elementary introduction to the surface differential geometry is given. The relations may be used as a common notation base for the geometric description of various shell models discussed in this Section.

Geometry of a Surface

A surface $\mathcal{M}$ in the three-dimensional Euclidean point space $\mathcal{E}$ can locally be defined by the position vector $\mathbf{r} = \mathbf{r}(\theta^\alpha)$ as the function of two curvilinear coordinates $\theta^\alpha, \alpha = 1,2$. Usually $\mathbf{r}$ is related to a reference frame $(O,\mathbf{i}_i)$, $i = 1,2,3$, in $\mathcal{E}$ with O a reference point and $\mathbf{i}_i$ some orthonormal vectors.

Two surface (covariant) vectors $\boldsymbol{a}_\alpha$ and the unit normal vector $\boldsymbol{n}$ defined by

$$\boldsymbol{a}_1 = \frac{\partial \mathbf{r}}{\partial \theta^1} \equiv \mathbf{r},_1 , \quad \boldsymbol{a}_2 = \frac{\partial \mathbf{r}}{\partial \theta^2} \equiv \mathbf{r},_2 , \quad \boldsymbol{n} = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|} , \tag{1}$$

form the fundamental triad of base vectors on $\mathcal{M}$. Two dual (contravariant) surface vectors are related to $\boldsymbol{a}_\alpha$ by

$$\boldsymbol{a}^\beta \cdot \boldsymbol{a}_\alpha = \delta_\alpha^\beta = \begin{cases} 1\, if \;\; \alpha = \beta, \\ 0\, if \;\; \alpha \neq \beta, \end{cases} \tag{2}$$

$$\boldsymbol{a}^1 = \frac{\boldsymbol{a}_2 \times \boldsymbol{n}}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)\cdot \boldsymbol{n}}, \quad \boldsymbol{a}^2 = \frac{\boldsymbol{n} \times \boldsymbol{a}_1}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)\cdot \boldsymbol{n}}. \tag{3}$$

The coefficients defined by

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta \tag{4}$$

are known as the covariant and contravariant components of the surface metric tensor. They allow one to calculate lengths of curves on $\mathcal{M}$, angles between them and areas on $\mathcal{M}$.

The following relations are satisfied:

$$\boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta, \quad \boldsymbol{a}_\alpha = a_{\alpha\beta}\boldsymbol{a}^\beta, \tag{5}$$

where the summation convention over the repeated Greek indices has been used. Similarly, the components $a^{\alpha\beta}$ and $a_{\alpha\beta}$ are used to raise or lower indices of components of the surface vectors and tensors.

In various geometric formulae it is convenient to make use of components of the surface alternation tensor

$$\begin{gathered} \varepsilon_{\alpha\beta} = (\boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta)\cdot \boldsymbol{n}, \quad \varepsilon^{\alpha\beta} = (\boldsymbol{a}^\alpha \times \boldsymbol{a}^\beta)\cdot \boldsymbol{n}, \\ \left[(\boldsymbol{a}_1 \times \boldsymbol{a}_2)\cdot \boldsymbol{n}\right]^2 = \det(a_{\alpha\beta}) = a > 0, \end{gathered} \tag{6}$$

which satisfy the relations

$$\begin{gathered} \varepsilon_{12} = -\varepsilon_{21} = \sqrt{a}, \quad \varepsilon_{11} = \varepsilon_{22} = 0, \\ \varepsilon^{12} = -\varepsilon^{21} = \frac{1}{\sqrt{a}}, \quad \varepsilon^{11} = \varepsilon^{22} = 0, \end{gathered} \tag{7}$$

$$\begin{gathered} \varepsilon_{\alpha\beta}\varepsilon^{\lambda\mu} = \delta_\alpha^\lambda \delta_\beta^\mu - \delta_\beta^\lambda \delta_\alpha^\mu, \quad \varepsilon_{\alpha\beta}\varepsilon^{\lambda\beta} = \delta_\alpha^\lambda, \quad \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} = 2, \\ \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}a_{\alpha\beta} = a^{\lambda\mu}, \quad a^{\alpha\lambda}a^{\beta\mu}\varepsilon_{\alpha\beta} = \varepsilon^{\lambda\mu}, \end{gathered} \tag{8}$$

$$\begin{gathered} \boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta = \varepsilon_{\alpha\beta}\boldsymbol{n}, \quad \boldsymbol{a}^\alpha \times \boldsymbol{a}^\beta = \varepsilon^{\alpha\beta}\boldsymbol{n}, \\ \boldsymbol{n} \times \boldsymbol{a}_\alpha = \varepsilon_{\alpha\beta}\boldsymbol{a}^\beta, \quad \boldsymbol{n} \times \boldsymbol{a}^\alpha = \varepsilon^{\alpha\beta}\boldsymbol{a}_\beta. \end{gathered} \tag{9}$$

Differentiation of the unit normal $\boldsymbol{n}$ with respect to the surface coordinates gives two vectors $\boldsymbol{n},_\alpha$ tangent to $\mathcal{M}$. The coefficients

$$b_{\alpha\beta} = -\boldsymbol{n},_\alpha \cdot \boldsymbol{a}_\beta = -\boldsymbol{n},_\beta \cdot \boldsymbol{a}_\alpha = \boldsymbol{n} \cdot \boldsymbol{a}_\alpha,_\beta \tag{10}$$

are known as the covariant components of the surface curvature tensor. Associated with $b_{\alpha\beta}$ two invariants

$$H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta}\,, \quad K = \frac{1}{2} \varepsilon^{\alpha\lambda} \varepsilon^{\beta\mu} b_{\alpha\beta} b_{\lambda\mu} \tag{11}$$

are called the mean and the Gaussian curvatures of $\mathcal{M}$, respectively. The components of $b_{\alpha\beta}$ allow one to calculate curvatures and torsions of the surface curves.

Covariant Differentiation

Partial differentiation of the base vectors $\boldsymbol{a}_\alpha$ with respect to the surface coordinates leads to

$$\boldsymbol{a}_\alpha,_\beta = \mathbf{r},_{\alpha\beta} = \Gamma_{\lambda.\alpha\beta} \boldsymbol{a}^\lambda + b_{\alpha\beta} \boldsymbol{n}\,, \tag{12}$$

where

$$\begin{aligned} \Gamma_{\lambda.\alpha\beta} &= \boldsymbol{a}_\lambda \cdot \boldsymbol{a}_\alpha,_\beta = \frac{1}{2}\left(a_{\lambda\alpha},_\beta + a_{\lambda\beta},_\alpha - a_{\alpha\beta},_\lambda \right), \\ \Gamma^\mu_{\alpha\beta} &= a^{\lambda\mu} \Gamma_{\lambda.\alpha\beta} = \boldsymbol{a}^\mu \cdot \boldsymbol{a}_\alpha,_\beta = -\boldsymbol{a}^\mu,_\alpha \cdot \boldsymbol{a}_\beta \end{aligned} \tag{13}$$

are called the surface Christoffel symbols of the first and second kind, respectively.

Differentiation of the surface tangent field $\boldsymbol{v} = v^\alpha \boldsymbol{a}_\alpha = v_\alpha \boldsymbol{a}^\alpha$ gives

$$\boldsymbol{v},_\beta = (v^\alpha \boldsymbol{a}_\alpha),_\beta = v^\alpha\,|_\beta\, \boldsymbol{a}_\alpha + b_{\alpha\beta} v^\alpha \boldsymbol{n} = v_{\alpha|\beta} \boldsymbol{a}^\alpha + b^\alpha_\beta v_\alpha \boldsymbol{n}\,, \tag{14}$$

where the operation $(.)_{|\alpha}$ defined by

$$v^\alpha\,|_\beta = v^\alpha,_\beta + \Gamma^\alpha_{\lambda\beta} v^\lambda\,, \quad v_{\alpha|\beta} = v_\alpha,_\beta - \Gamma^\lambda_{\alpha\beta} v_\lambda \tag{15}$$

is known as the covariant differentiation of the surface vector components.

Similarly, the covariant differentiation of the tangent surface tensor components are defined by

$$\begin{aligned} T^{\alpha\beta}\,|_\lambda &= T^{\alpha\beta},_\lambda + \Gamma^\alpha_{\kappa\lambda} T^{\kappa\beta} + \Gamma^\beta_{\kappa\lambda} T^{\alpha\kappa}\,, \\ T_{\alpha\beta|\lambda} &= T_{\alpha\beta},_\lambda - \Gamma^\kappa_{\alpha\lambda} T_{\kappa\beta} - \Gamma^\kappa_{\beta\lambda} T_{\alpha\kappa}\,. \end{aligned} \tag{16}$$

In particular, one can prove that

$$a_{\alpha\beta|\lambda} = a^{\alpha\beta}\,|_\lambda = \varepsilon_{\alpha\beta|\lambda} = \varepsilon^{\alpha\beta}\,|_\lambda = 0\,. \tag{17}$$

In case of the spatial vector field $\boldsymbol{w}$ defined in the surface bases by

$$\boldsymbol{w} = w^\alpha \boldsymbol{a}_\alpha + w\boldsymbol{n} = w_\alpha \boldsymbol{a}^\alpha + w\boldsymbol{n}\,, \tag{18}$$

its differentiation on $\mathcal{M}$ leads to

$$\begin{aligned} \boldsymbol{w},_\beta &= \left(w^\alpha\,|_\beta - b^\alpha_\beta w \right) \boldsymbol{a}_\alpha + \left(w,_\beta + b_{\alpha\beta} w^\alpha \right) \boldsymbol{n} \\ &= \left(w_{\alpha|\beta} - b_{\alpha\beta} w \right) \boldsymbol{a}^\alpha + \left(w,_\beta + b^\alpha_\beta w_\alpha \right) \boldsymbol{n}\,. \end{aligned} \tag{19}$$

Since $\boldsymbol{a}_{\beta|\lambda} = b_{\beta\lambda} \boldsymbol{n}$, $\boldsymbol{n},_\mu = \boldsymbol{n}_{|\mu} = -b^\kappa_\mu \boldsymbol{a}_\kappa$, the repeated covariant differentiation of $\boldsymbol{a}_\beta$ gives the vector identity

$$\boldsymbol{a}_{\beta|\lambda\mu} - \boldsymbol{a}_{\beta|\mu\lambda} = R_{\alpha\beta\lambda\mu}\boldsymbol{a}^{\alpha} \ , \tag{20}$$

or three scalar identities

$$b_{\beta\lambda|\mu} = b_{\beta\mu|\lambda} \ , \quad b_{\alpha\lambda}b_{\beta\mu} - b_{\alpha\mu}b_{\beta\lambda} = R_{\alpha\beta\lambda\mu} \ , \tag{21}$$

where $R_{\alpha\beta\lambda\mu}$ are components of the surface Riemann-Christoffel tensor which are defined entirely through the metric components by

$$R_{\alpha\beta\lambda\mu} = \frac{1}{2}\left(a_{\alpha\mu},_{\beta\lambda} + a_{\beta\lambda},_{\alpha\mu} - a_{\alpha\lambda},_{\beta\mu} - a_{\beta\mu},_{\alpha\lambda}\right) + \Gamma^{\kappa}_{\alpha\mu}\Gamma_{\kappa.\beta\lambda} - \Gamma^{\kappa}_{\alpha\lambda}\Gamma_{\kappa.\beta\mu} \ . \tag{0.22}$$

The components $R_{\alpha\beta\lambda\mu}$ have the following symmetry conditions:

$$R_{\alpha\beta\lambda\mu} = -R_{\beta\alpha\lambda\mu} = -R_{\alpha\beta\mu\lambda} = R_{\lambda\mu\alpha\beta} \ , \tag{0.23}$$

so that, in fact, they are expressible in terms of only one independent component R_{1212} .

The relations (21) are known as the Codazzi-Gauss equations for the surface. Since $\mathcal{M}$ in the space $\mathcal{E}$ has been described by three spatial components of $\mathbf{r}$, the relations (21) express just three compatibility conditions which must be satisfied by six components of $a_{\alpha\beta}$ and $b_{\alpha\beta}$.

Surface Curves

Let $\theta^{\alpha} = \theta^{\alpha}(s)$ define a curve $\mathcal{C}$ on the surface $\mathcal{M}$, where s is the length coordinate along $\mathcal{C}$. A special case of $\mathcal{C}$ is the surface curve $\partial\mathcal{M}$ describing an edge of $\mathcal{M}$. With each point of $\mathcal{C}$ one can associate the orthonormal triad of vectors $\boldsymbol{\tau},\boldsymbol{n},\boldsymbol{\nu}$, where $\boldsymbol{\tau}$ is the unit vector tangent to $\mathcal{C}$ and $\boldsymbol{\nu}$ is the outward unit normal vector tangent to $\mathcal{M}$ defined by

$$\begin{aligned} &\boldsymbol{\tau} = \mathbf{r},_s = \tau_{\alpha}\boldsymbol{a}^{\alpha} \ , \quad \boldsymbol{\nu} = \boldsymbol{\tau}\times\boldsymbol{n} = \nu_{\alpha}\boldsymbol{a}^{\alpha} \ , \\ &\boldsymbol{a}_{\alpha} = \nu_{\alpha}\boldsymbol{\nu} + \tau_{\alpha}\boldsymbol{\tau} \ , \quad \nu^{\beta} = \varepsilon^{\beta\alpha}\tau_{\alpha} \ , \quad \tau^{\beta} = \varepsilon^{\alpha\beta}\nu_{\alpha} \ . \end{aligned} \tag{24}$$

Differentiation of the triad $\boldsymbol{\nu},\boldsymbol{\tau},\boldsymbol{n}$ along $\mathcal{C}$ gives

$$\frac{d}{ds}\begin{bmatrix}\boldsymbol{\nu}\\ \boldsymbol{\tau}\\ \boldsymbol{n}\end{bmatrix} = \boldsymbol{\omega}_{\tau}\times\begin{bmatrix}\boldsymbol{\nu}\\ \boldsymbol{\tau}\\ \boldsymbol{n}\end{bmatrix}, \quad \boldsymbol{\omega}_{\tau} = \sigma_{\tau}\boldsymbol{\nu} + \tau_{\tau}\boldsymbol{\tau} + \rho_{\tau}\boldsymbol{n} \ , \tag{25}$$

where the normal curvature σ_{τ}, the geodesic torsion τ_{τ} and the geodesic curvature ρ_{τ} of the surface curve $\mathcal{C}$ are defined by

$$\sigma_{\tau} = b_{\alpha\beta}\tau^{\alpha}\tau^{\beta} \ , \quad \tau_{\tau} = -b_{\alpha\beta}\nu^{\alpha}\tau^{\beta} \ , \quad \rho_{\tau} = \tau_{\alpha}\nu^{\alpha}\,|_{\beta}\,\tau^{\beta} \ . \tag{26}$$

The value of σ_{τ} at each point $M \in \mathcal{M}$ depends on the direction $\boldsymbol{\tau}$ of $\mathcal{C}$. The principal directions are those for which $b_{\alpha\beta}\tau^{\alpha}\tau^{\beta}$ assume the extreme values under the condition $a_{\alpha\beta}\tau^{\alpha}\tau^{\beta} = 1$. The problem is equivalent to finding extremal values of the function

$$F\left(\tau^{\alpha}\right) = b_{\alpha\beta}\tau^{\alpha}\tau^{\beta} - \sigma_{\tau}\left(a_{\alpha\beta}\tau^{\alpha}\tau^{\beta} - 1\right) \ . \tag{27}$$

Differentiation of (27) leads to the set of two homogeneous algebraic equations

$$\frac{1}{2}\frac{\partial F}{\partial \tau^{\alpha}} = \left(b_{\alpha\beta} - \sigma_{\tau} a_{\alpha\beta}\right)\tau^{\beta} = 0\,, \tag{28}$$

which non-trivial solutions exist provided that

$$\det\left(b_{\alpha\beta} - \sigma_{\tau} a_{\alpha\beta}\right) = 0\,, \quad \sigma_{\tau}^{2} - 2\sigma_{\tau} H + K = 0\,. \tag{29}$$

The roots of $(29)_2$ are

$$\sigma_{1,2} = H \pm \sqrt{H^{2} - K} \tag{30}$$

and are called the surface principal normal curvatures at the point $M \in \mathcal{M}$.

The values σ_1 and σ_2 in (30) are always real, because

$$4\left(H^{2} - K\right) = \left(b_1^1 + b_2^2\right)^2 + 4\left(b_2^1\right)^2 \geq 0\,. \tag{31}$$

The equality in (31) takes place only if $b_1^1 = b_2^2$ and $b_1^2 = b_2^1$, under which the point of $\mathcal{M}$ is called spherical. If a point $M \in \mathcal{M}$ is not spherical then σ_1 and σ_2 assume different values such that

$$\sigma_1 \sigma_2 = K\,, \quad \sigma_1 + \sigma_2 = 2H\,. \tag{32}$$

The point $M \in \mathcal{M}$ is called elliptic when $K > 0$, parabolic when $K = 0$ and hyperbolic when $K < 0$. The surface consisting entirely of one type of points mentioned above is called the surface of positive, zeroth or negative Gaussian curvature, respectively.

The curve on a surface having tangent at each point along the principal direction is called the line of principal curvature. If $\tau_{(1)}^{\alpha}$ and $\tau_{(2)}^{\alpha}$ are two principal directions corresponding to σ_1 and σ_2, respectively, then multiplying (28) in succession by $\tau_{(1)}^{\alpha}$ and $\tau_{(2)}^{\alpha}$, and subtracting the results one obtains

$$\left(\sigma_2 - \sigma_1\right) a_{\alpha\beta} \tau_{(1)}^{\alpha} \tau_{(2)}^{\beta} = \left(\sigma_2 - \sigma_1\right) \boldsymbol{\tau}_{(1)} \cdot \boldsymbol{\tau}_{(2)} = 0\,. \tag{33}$$

This means that, unless the point is spherical, the two principal directions are orthogonal. Moreover, multiplying the conditions (28) by ν^{α} one finds that in the principal directions $b_{\alpha\beta}\tau^{\beta}\nu^{\alpha} = 0$, so that the geodesic torsion τ_{τ} vanishes identically along the principal line.

Various surfaces may conveniently be parameterized by different surface coordinate systems, which may neither be orthogonal nor coinciding with lines of principal curvatures, in general. But majority of surfaces are described and analysed in orthogonal coordinates coinciding with lines of principal curvatures. In this case it is convenient to introduce the unit surface base vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ such that

$$\begin{gathered} \boldsymbol{a}_1 = A_1 \boldsymbol{e}_1\,, \quad \boldsymbol{a}_2 = A_2 \boldsymbol{e}_2\,, \quad A_1 = \sqrt{a_{11}}\,, \quad A_2 = \sqrt{a_{22}}\,, \\ a^{11} = \frac{1}{\left(A_1\right)^2}\,, \quad a^{22} = \frac{1}{\left(A_2\right)^2}\,, \quad a^{12} = a_{12} = 0\,, \quad a = \left(A_1 A_2\right)^2\,. \end{gathered} \tag{34}$$

For orthogonal surface coordinates of principal curvatures, the non-zero Christoffel symbols of the second kind one finds from (13) to be

$$\begin{gathered}\Gamma^1_{11} = \frac{1}{A_1}\frac{\partial A_1}{\partial \theta^1}, \quad \Gamma^1_{22} = -\frac{A_2}{(A_1)^2}\frac{\partial A_2}{\partial \theta^1}, \quad \Gamma^1_{12} = \frac{1}{A_1}\frac{\partial A_1}{\partial \theta^2}, \\ \Gamma^2_{22} = \frac{1}{A_2}\frac{\partial A_2}{\partial \theta^2}, \quad \Gamma^2_{11} = -\frac{A_1}{(A_2)^2}\frac{\partial A_1}{\partial \theta^2}, \quad \Gamma^2_{12} = \frac{1}{A_2}\frac{\partial A_2}{\partial \theta^1}.\end{gathered} \tag{35}$$

The symbols (35) allow one to express covariant differentiation of vector and tensor components in terms of their partial differentiation.

With the help of (34) any vector $\boldsymbol{w}$ can be represented by its components as

$$\boldsymbol{w} = w_{<1>}\boldsymbol{e}_1 + w_{<2>}\boldsymbol{e}_2 + w\boldsymbol{n}, \quad w_{<\alpha>} = A_\alpha w^\alpha = \frac{1}{A_\alpha} w_\alpha \text{ (no sum over } \alpha). \tag{36}$$

The components $w_{<\alpha>}$ are called the physical components of the vector $\boldsymbol{w}$.

Similarly, one can introduce the physical components of the surface tensor by

$$T_{<\alpha\beta>} = \frac{1}{A_\alpha A_\beta} T_{\alpha\beta} = A_\alpha A_\beta T^{\alpha\beta} = \frac{A_\beta}{A_\alpha} T_\alpha^{\cdot\beta} \text{ (no sum over } \alpha,\beta). \tag{37}$$

In particular, the physical components of the surface curvature tensor $b_{\alpha\beta}$ are defined by

$$b_{<11>} = \frac{b_{11}}{(A_1)^2} = -\frac{1}{R_1}, \quad b_{<22>} = \frac{b_{22}}{(A_2)^2} = -\frac{1}{R_2}, \tag{38}$$

where R_1 and R_2 are the principal radii of curvatures of corresponding lines of principal curvatures.

Surface Divergence Theorem

Let the regular surface $\mathcal{M}$ be bounded by a closed smooth boundary $\partial\mathcal{M}$. The divergence of a tangential vector field $\boldsymbol{w} = w^\alpha \boldsymbol{a}_\alpha$ defined over $\mathcal{M}$ is the scalar field defined by $\mathrm{div}_s\, \boldsymbol{w} = \boldsymbol{w},_\alpha \cdot \boldsymbol{a}^\alpha = w^\alpha|_\alpha$. For a mixed 2nd-order tensor field on $\mathcal{M}$, $\boldsymbol{S} = \boldsymbol{S}^\alpha \otimes \boldsymbol{a}_\alpha$, $\boldsymbol{S}^\alpha = S^{i\alpha}\boldsymbol{c}_i$, where $\boldsymbol{c}_i, i = 1,2,3,$ is any 3D vector base on $\mathcal{M}$ and $\otimes$ is the tensor product, the surface divergence is the vector field defined by $\mathrm{div}_s\, \boldsymbol{S} = \boldsymbol{S}^\alpha|_\alpha$.

The surface divergence theorem (also called the Green or Gauss theorem) relates integrals of the fields $\boldsymbol{w}$ and $\boldsymbol{S}$ along the boundary $\partial\mathcal{M}$ to their divergences over the surface $\mathcal{M}$ according to

$$\begin{gathered}\int_{\partial\mathcal{M}} w^\alpha \nu_\alpha ds = \iint_{\mathcal{M}} w^\alpha|_\alpha \, da, \\ \int_{\partial\mathcal{M}} \boldsymbol{S}^\alpha \nu_\alpha ds = \iint_{\mathcal{M}} \boldsymbol{S}^\alpha|_\alpha \, da.\end{gathered} \tag{39}$$

In shell theory these theorems are used to derive the Euler equations of some variational statements.

Cross References

Thin Elastic Shells, Linear Theory
Thin Elastic Shells, Lagrangian Geometrically Non-linear Theory
Elastic Shells, Resultant Non-linear Theory
Junctions in Irregular Shell Structures

References

Başar Y, Krätzig WB (2001) Theory of shell structures, 2nd ed. VDI Verlag, Düsseldorf

Chernykh KF (1964) Linear theory of shells, Part 2 (in Russian). University Press, Leningrad. English translation: NASA-TT-F-II 562, 1968

Ciarlet PG (2005) An introduction to differential geometry with application to elasticity. Springer, Berlin

Do Carmo MP (1976) Differential geometry of curves and surfaces. Prentice Hall, New Jersey

Eisenhart LP (1947) An introduction to differential geometry with use of the tensor calculus. University Press, Princeton

Flügge W (1972) Tensor analysis and continuum mechanics. Springer-Verlag, Berlin et al.

Green AE, Zerna W (1954) Theoretical elasticity. Clarendon Press, Oxford

Naghdi PM (1963) Foundations of elastic shell theory. In: Sneddon IN, Hill R (eds), Progress in Solid Mechanics IV. North-Holland, Amsterdam, pp 1-90

Pietraszkiewicz W (1977) Introduction to the non-linear theory of shells. Ruhr-Universität, Mitt Inst für Mech 10, Bochum

Lebedev LP, Cloud MJ, Eremeyev VA (2010) Tensor analysis with applications in mechanics. World Scientific, Singapore