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Thin Elastic Shells, Lagrangian Geometrically Non-linear Theory

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Synonyms

Kirchhoff-Love non-linear shell model; Small-strain thin shell theory

Definition

The Lagrangian non-linear theory of thin elastic shells is an approximate two-dimensional special case of geometrically non-linear theory of elasticity. Here the shell thickness is supposed to be much smaller than the smallest radius of curvature of the undeformed shell middle surface. As a result, the shell deformation can approximately be described only by stretching and bending of its middle surface. As compared with the classical linear theory of thin elastic shells discussed in another entry in this Encyclopaedia (Pietraszkiewicz 2018b), here only strains in the shell space are assumed to be small, while rotations of material elements are initially not restricted. The Lagrangian shell theory mean that all shell relations are formulated entirely in the undeformed geometry of the shell midsurface.

Introduction

Due to non-linearity of deformation, invariant non-linear relations of thin elastic shells can be formulated either in Eulerian, or in Lagrangian, or in several mixed descriptions (Pietraszkiewicz 1989). The Lagrangian thin shell relations are formulated entirely in the geometry of undeformed shell midsurface. The Lagrangian shell relations are of primary importance for most applications, because the undeformed shell midsurface is usually the only one known in advance.

The quite general Lagrangian thin shell relations were proposed already by Galimov (1951) applying two steps. First, the corresponding simple Eulerian equilibrium equations and boundary conditions were given in the unknown deformed midsurface base. Then appropriate transformation rules were applied to express the vectorial Eulerian quantities in terms of the Lagrangian ones. Unfortunately, under such transformations the fourth scalar (rotational) boundary condition was still referred to the deformed lateral boundary surface. Only much later it was proved (Makowski and Pietraszkiewicz 1989) that such a transformation of the fourth boundary condition does not lead, in general, to a fourth Lagrangian boundary condition, because some differential one-form associated with the virtual rotation parameter referred to the deformed boundary is not integrable in terms of translation surface derivatives. This was the reason why no variational principles could be constructed for such quasi-Lagrangian shell theory even for conservative surface and boundary loadings.

The entirely Lagrangian thin shell relations were worked out by Pietraszkiewicz and Szwabowicz (1981) using a modified tensor of change of curvature. These relations were reworked for the classical tensor of change of curvature (Pietraszkiewicz 1984) together with several consistently simplified versions of shell relations under small strains and restricted rotations as well as with two incremental formulations of the relations in the total Lagrangian and updated Lagrangian descriptions.

In this note the entirely Lagrangian non-linear theory of thin elastic shells is outlined. It is based on the following three assumptions:

1. The strains in the shell space are small, but rotations of material elements are initially not restricted.
2. The material elements, which are normal to the shell middle surface in the undeformed placement, remain normal to the deformed shell midsurface and do not change their lengths. This assumption allows one to approximately describe the non-linear shell deformation only by stretching and bending of its middle surface.
3. The state of elastic stress in the shell space is approximately plane. This means that the effects of transverse shear stresses and of normal stresses, acting on surfaces parallel to the middle surface, may be neglected in the elastic strain energy density.

To be concise, the assumptions 2. and 3. are used to derive the approximate equilibrium equations and boundary conditions from the postulated principle of virtual displacements for the shell midsurface. The resulting shell relations are initially formulated for unrestricted surface deformation measures and unrestricted displacements. Then consistently approximated relations under small elastic strains are discussed. Finally, several simplified sets of shell relations under additional consistently restricted rotations of material elements are given.

Geometry and Deformation of the Shell Middle Surface

Let $\mathcal{P}$ be the region of three-dimensional Euclidean point space $\mathcal{E}$ occupied by the shell in the undeformed placement. The position vector of any point $\mathrm{P} \in \mathcal{P}$ relative to a reference point $\mathrm{O} \in \mathcal{E}$ can be given by

$$\mathbf{p}(\theta^\alpha, \xi) = \mathrm{p}^i(\theta^\alpha)\mathbf{i}_i = \boldsymbol{r}(\theta^\alpha) + \xi \boldsymbol{n}(\theta^\alpha)\,, \tag{1}$$

where $\mathbf{i}_i,\ i = 1,2,3,$ are three orthonormal vectors of a reference frame, $\theta^\alpha,\ \alpha = 1,2,$ are the curvilinear surface coordinates, $-h/2 \le \xi \le +h/2$ is the distance from the shell midsurface $\mathcal{M}$ defined by the position vector $\boldsymbol{r}(\theta^\alpha)$, $\boldsymbol{n}(\theta^\alpha)$ is the unit normal vector orienting $\mathcal{M}$, and h is the shell thickness. In thin shell theory it is understood that h be constant and small as compared with the smallest radius of curvature R of $\mathcal{M}$ and with linear dimensions of $\mathcal{P}$.

Geometry of the middle surface $\mathcal{M}$ is described by the following fields (Pietraszkiewicz 2018a): the natural base vectors $\boldsymbol{a}_\alpha = \partial \boldsymbol{r} / \partial \theta^\alpha \equiv \boldsymbol{r}_{,\alpha}$, the covariant components $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ of the surface metric tensor with determinant $a = \det(a_{\alpha\beta})$, the dual (contravariant) base vectors $\boldsymbol{a}^\beta$ such that $\boldsymbol{a}_\alpha \cdot \boldsymbol{a}^\beta = \delta_\alpha^\beta$, where $\delta_1^1 = \delta_2^2 = 1,\ \delta_2^1 = \delta_1^2 = 0$, the unit normal vector $\boldsymbol{n} = \boldsymbol{a}_1 \times \boldsymbol{a}_2 / |\boldsymbol{a}_1 \times \boldsymbol{a}_2|$, and the covariant components $b_{\alpha\beta} = -\boldsymbol{a}_\alpha \cdot \boldsymbol{n}_{,\beta} = -\boldsymbol{n}_{,\alpha} \cdot \boldsymbol{a}_\beta$ of the surface curvature tensor. The contravariant metric components $a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$ are used to raise indices of surface vectors and tensors, for example $\boldsymbol{a}^\beta = a^{\beta\alpha}\boldsymbol{a}_\alpha$, $b^\alpha_\beta = a^{\alpha\lambda} b_{\lambda\beta}$ etc., where the summation convention over the repeated indices is used.

The shell middle surface in the deformed placement $\bar{\mathcal{M}}$ is described by the position vector $\bar{\boldsymbol{r}}(\theta^\alpha) = \boldsymbol{r}(\theta^\alpha) + \boldsymbol{u}(\theta^\alpha)$, where θ^α are the surface curvilinear convective coordinates and $\boldsymbol{u} = u_\alpha \boldsymbol{a}^\alpha + w\boldsymbol{n}$ is the translation vector field. The geometric quantities describing $\bar{\mathcal{M}}$ are analogous to those describing $\mathcal{M}$, only now they are marked by the overbar: $\bar{\boldsymbol{a}}_\alpha = \bar{\boldsymbol{r}},_\alpha$, $\bar{a}_{\alpha\beta} = \bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{a}}_\beta$, $\bar{a} = \det(\bar{a}_{\alpha\beta}) > 0$, $\bar{\boldsymbol{a}}^\beta \cdot \bar{\boldsymbol{a}}_\alpha = \delta^\beta_\alpha$, $\bar{\boldsymbol{n}} = \bar{\boldsymbol{a}}_1 \times \bar{\boldsymbol{a}}_2 / |\bar{\boldsymbol{a}}_1 \times \bar{\boldsymbol{a}}_2|$, $\bar{b}_{\alpha\beta} = -\bar{\boldsymbol{n}},_\alpha \cdot \bar{\boldsymbol{a}}_\beta = -\bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{n}},_\beta$, $\bar{a}^{\alpha\beta} = \bar{\boldsymbol{a}}^\alpha \cdot \bar{\boldsymbol{a}}^\beta$ etc. The barred quantities can be expressed through analogous unbarred ones and the vector $\boldsymbol{u}$ by the relations (Pietraszkiewicz 1980, 1989)

$$\begin{aligned} \bar{\boldsymbol{a}}_\alpha &= \boldsymbol{a}_\alpha + \boldsymbol{u},_\alpha = l^\lambda_{\cdot\alpha}\boldsymbol{a}_\lambda + \varphi_\alpha \boldsymbol{n}, \\ \bar{\boldsymbol{n}} &= \frac{1}{2j}\varepsilon^{\alpha\beta}\bar{\boldsymbol{a}}_\alpha \times \bar{\boldsymbol{a}}_\beta = n^\lambda \boldsymbol{a}_\lambda + n\boldsymbol{n}, \end{aligned} \tag{2}$$

where

$$l_{\alpha\beta} = a_{\alpha\beta} + u_{\alpha|\beta} - b_{\alpha\beta}w, \quad \varphi_\alpha = w,_\alpha + b^\lambda_\alpha u_\lambda, \tag{3}$$

$$\begin{aligned} &\varepsilon^{\alpha\beta} = \left(\boldsymbol{a}^\alpha \times \boldsymbol{a}^\beta\right)\cdot \boldsymbol{n}, \quad j = \sqrt{\frac{\bar{a}}{a}}, \\ &n_\mu = \frac{1}{j}\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}\varphi_\alpha l^\lambda_{\cdot\beta}, \quad n = \frac{1}{2j}\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu} l^\lambda_{\cdot\alpha} l^\mu_{\cdot\beta}. \end{aligned} \tag{4}$$

The metric and curvature tensor components of $\bar{\mathcal{M}}$ are given by

$$\bar{a}_{\alpha\beta} = a_{\alpha\beta} + 2\gamma_{\alpha\beta}, \quad \bar{b}_{\alpha\beta} = b_{\alpha\beta} - \kappa_{\alpha\beta}, \tag{5}$$

$$\begin{aligned} \gamma_{\alpha\beta} &= \frac{1}{2}\left(\bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{a}}_\beta - a_{\alpha\beta}\right) = \frac{1}{2}\left(l^\lambda_{\cdot\alpha} l_{\lambda\beta} + \varphi_\alpha\varphi_\beta - a_{\alpha\beta}\right), \\ \kappa_{\alpha\beta} &= \bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{n}},_\beta + b_{\alpha\beta} = l_{\lambda\alpha}\left(n^\lambda|_\beta - b^\lambda_\beta n\right) + \varphi_\alpha\left(n,_\beta + b^\lambda_\beta n_\lambda\right) + b_{\alpha\beta}. \end{aligned} \tag{6}$$

Here $\gamma_{\alpha\beta}$ are the Lagrangian components of the surface strains while $\kappa_{\alpha\beta}$ are the Lagrangian components of the surface changes of curvatures (briefly bendings). Both surface deformation measures are symmetric: $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}$, $\kappa_{\alpha\beta} = \kappa_{\beta\alpha}$. The components $\gamma_{\alpha\beta}$ are quadratic polynomials of u_α, w and their first surface derivatives, while the $\kappa_{\alpha\beta}$ are non-rational functions of u_α, w and their first as well as second surface derivatives. The non-rationality of $\kappa_{\alpha\beta}$ is caused by the invariant j, where

$$j^2 = 1 + 2\gamma^\alpha_\alpha + 2\left(\gamma^\alpha_\alpha\gamma^\beta_\beta - \gamma^\alpha_\beta\gamma^\beta_\alpha\right). \tag{7}$$

The boundary contour $\partial\mathcal{M}$ of $\mathcal{M}$ consists of the finite set of piecewise smooth curves $\boldsymbol{r}(s) = \boldsymbol{r}[\theta^\alpha(s)]$, where s is the arc length along $\partial\mathcal{M}$. In each regular point $M \in \partial\mathcal{M}$ two unit vectors are defined: the tangent $\boldsymbol{\tau} = d\boldsymbol{r}/ds \equiv \boldsymbol{r}' = \boldsymbol{a}_\alpha \tau^\alpha$ and the outward normal

$\boldsymbol{\nu}=\boldsymbol{\tau}\times\boldsymbol{n}=\boldsymbol{a}_\alpha \nu^\alpha \equiv \boldsymbol{r},_\nu$, $\nu^\alpha=\varepsilon^{\alpha\beta}\tau_\beta$, where $\boldsymbol{r},_\nu$ means the outward normal derivative of $\boldsymbol{r}$ at $\partial\mathcal{M}$.

The position vector of the deformed boundary contour $\partial\bar{\mathcal{M}}$ is given by $\bar{\boldsymbol{r}}(s)=\boldsymbol{r}(s)+\boldsymbol{u}(s)$, where s is the convective coordinate. The following relations are satisfied along $\partial\bar{\mathcal{M}}$:

$$\bar{\boldsymbol{r}}\,'=\bar{\boldsymbol{a}}_\alpha\tau^\alpha=\boldsymbol{\tau}+\boldsymbol{u}'=a_\tau\bar{\boldsymbol{\tau}},\quad \bar{\boldsymbol{r}},_\nu=\bar{\boldsymbol{a}}_\alpha\nu^\alpha=\boldsymbol{\nu}+\boldsymbol{u},_\nu\,,\quad \bar{\boldsymbol{n}}=j^{-1}\bar{\boldsymbol{r}},_\nu\times\bar{\boldsymbol{r}}\,',\tag{8}$$

$$a_\tau=|\bar{\boldsymbol{r}}\,'|=\sqrt{1+2\boldsymbol{u}'\cdot\boldsymbol{\tau}+\boldsymbol{u}'\cdot\boldsymbol{u}'}\,,\quad j^2=|\bar{\boldsymbol{r}},_\nu|^2|\bar{\boldsymbol{r}}\,'|^2-|\bar{\boldsymbol{r}},_\nu\cdot\bar{\boldsymbol{r}}\,'|^2\,.\tag{9}$$

All the barred quantities appearing in (2) to (9) are understood to be expressed through components of the translation vector $\boldsymbol{u}$ and its surface derivatives in the undeformed bases $\boldsymbol{a}_\alpha,\boldsymbol{n}$ or $\boldsymbol{\nu},\boldsymbol{\tau},\boldsymbol{n}$, respectively.

Lagrangian Equilibrium Conditions

Let the shell be loaded by the surface force $\boldsymbol{f}=f^\alpha\boldsymbol{a}_\alpha+f\boldsymbol{n}$ and the surface couple $\boldsymbol{m}=\bar{\boldsymbol{n}}\times(m^\alpha\bar{\boldsymbol{a}}_\alpha)$ vector fields, both measured per unit area of $\mathcal{M}$, as well as by the boundary force $\boldsymbol{n}^*=n_\nu^*\boldsymbol{\nu}+n_\tau^*\boldsymbol{\tau}+n^*\boldsymbol{n}$ and the boundary couple $\boldsymbol{m}^*=\bar{\boldsymbol{n}}\times\left(m_\nu^*\bar{\boldsymbol{\nu}}+m_\tau^*\bar{\boldsymbol{\tau}}\right)$ vectors, both given per unit length of $\partial\mathcal{M}$. Then for any additional kinematically admissible virtual translation $\delta\boldsymbol{u}=\delta u_\alpha\boldsymbol{a}^\alpha+\delta w\boldsymbol{n}$ the following Lagrangian principle of virtual displacements should be satisfied (Pietraszkiewicz 1984, 1989):

$$\begin{aligned}&\iint_{\mathcal{M}}\left(N^{\alpha\beta}\delta\gamma_{\alpha\beta}+M^{\alpha\beta}\delta\kappa_{\alpha\beta}\right)da\\ &\quad=\iint_{\mathcal{M}}\left(\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{m}\cdot\delta\boldsymbol{\omega}\right)da+\int_{\partial\mathcal{M}_f}\left(\boldsymbol{n}^*\cdot\delta\boldsymbol{u}+\boldsymbol{m}^*\cdot\delta\boldsymbol{\omega}_\tau\right)ds\,.\end{aligned}\tag{10}$$

In (10), $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are symmetric components of the internal 2nd Piola-Kirchhoff type surface stress and couple resultant tensors, $\delta\boldsymbol{\omega}$ and $\delta\boldsymbol{\omega}_\tau$ are the virtual rotation vectors on $\mathcal{M}$ and $\partial\mathcal{M}$, respectively, while the virtual surface deformation measures are given by

$$\begin{aligned}\delta\gamma_{\alpha\beta}&=\frac{1}{2}\left(\delta\boldsymbol{u},_\alpha\cdot\bar{\boldsymbol{a}}_\beta+\bar{\boldsymbol{a}}_\alpha\cdot\delta\boldsymbol{u},_\beta\right),\\ \delta\kappa_{\alpha\beta}&=\frac{1}{2}\left(\bar{\boldsymbol{n}},_\alpha\cdot\delta\boldsymbol{u},_\beta+\bar{\boldsymbol{n}},_\beta\cdot\delta\boldsymbol{u},_\alpha+\bar{\boldsymbol{a}}_\alpha\cdot\delta\bar{\boldsymbol{n}},_\beta+\bar{\boldsymbol{a}}_\beta\cdot\delta\bar{\boldsymbol{n}},_\alpha\right).\end{aligned}\tag{11}$$

Variating the three constraints $\bar{\boldsymbol{n}}\cdot\bar{\boldsymbol{a}}_\alpha=0$, $\bar{\boldsymbol{n}}\cdot\bar{\boldsymbol{n}}=1$ satisfied on $\bar{\mathcal{M}}$, one obtains $\delta\bar{\boldsymbol{n}}\cdot\bar{\boldsymbol{a}}_\alpha=-\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u},_\alpha$, $\delta\bar{\boldsymbol{n}}\cdot\bar{\boldsymbol{n}}=0$, so that $\delta\bar{\boldsymbol{n}}=-\bar{\boldsymbol{a}}^\alpha\left(\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u},_\alpha\right)$. On the other hand, $\delta\boldsymbol{\omega}$ on $\bar{\mathcal{M}}$ should satisfy $\delta\bar{\boldsymbol{n}}=\delta\boldsymbol{\omega}\times\bar{\boldsymbol{n}}$. As a result, in (10) one has $\boldsymbol{m}\cdot\delta\boldsymbol{\omega}=-m^\alpha\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u},_\alpha$ on $\mathcal{M}$.

Introducing (11) and the above relation for $\boldsymbol{m}\cdot\delta\boldsymbol{\omega}$ to (10) and applying the surface divergence theorem (Pietraszkiewicz 2018), one can transform (10) into

$$
\begin{aligned}
&-\iint_{\mathcal{M}}\left[\left(\boldsymbol{T}^{\alpha}+m^{\alpha}\bar{\boldsymbol{n}}\right)|_{\alpha}+\boldsymbol{f}\,\right]\cdot\delta\boldsymbol{u}\,da \\
&\quad+\int_{\partial\mathcal{M}_f}\left\{\left[\left(\boldsymbol{T}^{\alpha}+m^{\alpha}\bar{\boldsymbol{n}}\right)\nu_{\alpha}-\boldsymbol{n}^{*}\right]\cdot\delta\boldsymbol{u}+\left(M^{\alpha\beta}\nu_{\alpha}\bar{\boldsymbol{a}}_{\beta}\right)\cdot\delta\bar{\boldsymbol{n}}-\boldsymbol{m}^{*}\cdot\delta\boldsymbol{\omega}_{\tau}\right\}ds \\
&\quad+\int_{\partial\mathcal{M}_d}\left[\left(\boldsymbol{T}^{\alpha}+m^{\alpha}\bar{\boldsymbol{n}}\right)\nu_{\alpha}\cdot\delta\boldsymbol{u}+\left(M^{\alpha\beta}\nu_{\alpha}\bar{\boldsymbol{a}}_{\beta}\right)\cdot\delta\bar{\boldsymbol{n}}\right]ds=0\,,
\end{aligned}
\tag{12}
$$

where

$$
\boldsymbol{T}^{\alpha}=N^{\alpha\beta}\bar{\boldsymbol{a}}_{\beta}+M^{\alpha\beta}\bar{\boldsymbol{n}}_{,\beta}+\left[\left(M^{\kappa\rho}\bar{\boldsymbol{a}}_{\rho}\right)|_{\kappa}\cdot\bar{\boldsymbol{a}}^{\alpha}\right]\cdot\bar{\boldsymbol{n}}\,, \tag{13}
$$

and $\partial\mathcal{M}_d=\partial\mathcal{M}\setminus\partial\mathcal{M}_f$ is the complementary part of $\partial\mathcal{M}$ along which the kinematic boundary conditions are prescribed.

The vector $\bar{\boldsymbol{n}}(s)$ along $\partial\mathcal{M}$ satisfies only two constraints $\bar{\boldsymbol{r}}'\cdot\bar{\boldsymbol{n}}=0$, $\bar{\boldsymbol{n}}\cdot\bar{\boldsymbol{n}}=1$. This means that in order to establish the unique position of $\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\tau}},\bar{\boldsymbol{n}}$ relative to $\boldsymbol{\nu},\boldsymbol{\tau},\boldsymbol{n}$ one has to know not only three components of $\boldsymbol{u}(s)$ (thus also $\boldsymbol{u}'(s)$) but additionally one scalar function $\varphi(s)=\varphi[\boldsymbol{u}_{,\nu}(s),\boldsymbol{u}'(s)]$ describing the rotational deformation between the bases. The meaning of $\varphi(\boldsymbol{u}_{,\nu},\boldsymbol{u}')$ is not unique and depends on how the rotational deformation is defined.

The structure of the Lagrangian boundary conditions along $\partial\mathcal{M}$ has been discussed by Makowski and Pietraszkiewicz (1989) with the help of integrability conditions of some differential one-forms. It has been found, in particular, that the general relation for $\delta\bar{\boldsymbol{n}}(\varphi,\boldsymbol{u}')$ at the boundary contour can be given in the form

$$
\delta\bar{\boldsymbol{n}}=\boldsymbol{q}\,\delta\varphi+\boldsymbol{L}\delta\boldsymbol{u}'\,. \tag{14}
$$

Since $\delta\bar{\boldsymbol{n}}=\delta\boldsymbol{\omega}_{\tau}\times\bar{\boldsymbol{n}}$ along the boundary contour, one obtains $\boldsymbol{m}^{*}\cdot\delta\boldsymbol{\omega}_{\tau}=\left(\boldsymbol{m}^{*}\times\bar{\boldsymbol{n}}\right)\cdot\delta\bar{\boldsymbol{n}}$. Introducing this relation together with (14) into the curvilinear integrals of (12), after integration by parts one can transform them into

$$
\begin{aligned}
&\int_{\partial\mathcal{M}_f}\left[\left(\boldsymbol{P}+m_{\nu}\bar{\boldsymbol{n}}-\boldsymbol{P}^{*}\right)\cdot\delta\boldsymbol{u}+\left(M-M^{*}\right)\delta\varphi\right]ds+\sum_{M_n\in\partial\mathcal{M}_f}\left(\boldsymbol{F}_n-\boldsymbol{F}_n^{*}\right)\cdot\delta\boldsymbol{u}_n \\
&\quad+\int_{\partial\mathcal{M}_d}\left[\left(\boldsymbol{P}+m_{\nu}\bar{\boldsymbol{n}}\right)\cdot\delta\boldsymbol{u}\right]ds+\sum_{M_n\in\partial\mathcal{M}_d}\boldsymbol{F}_n\cdot\delta\boldsymbol{u}_n=0\,,
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
&\boldsymbol{F}=-\boldsymbol{L}^{T}\left(M^{\alpha\beta}\bar{\boldsymbol{a}}_{\beta}\nu_{\alpha}\right),\quad \boldsymbol{F}^{*}=-\boldsymbol{L}^{T}\left(\boldsymbol{m}^{*}\times\bar{\boldsymbol{n}}\right), \\
&\boldsymbol{P}=\boldsymbol{T}^{\alpha}\nu_{\alpha}+\boldsymbol{F}'\,,\quad \boldsymbol{P}^{*}=\boldsymbol{n}^{*}+\boldsymbol{F}^{*}{}'\,,\quad m_{\nu}=m^{\alpha}\nu_{\alpha}\,, \\
&M=\boldsymbol{q}\cdot\left(M^{\alpha\beta}\bar{\boldsymbol{a}}_{\beta}\nu_{\alpha}\right),\quad M^{*}=\boldsymbol{q}\cdot\left(\boldsymbol{m}^{*}\times\bar{\boldsymbol{n}}\right), \\
&\boldsymbol{F}_n=\boldsymbol{F}_n\left(s_n+0\right)-\boldsymbol{F}_n\left(s_n-0\right),\quad \boldsymbol{u}_n=\boldsymbol{u}(s_n)\,.
\end{aligned}
\tag{16}
$$

Kinematically admissible virtual displacements satisfy $\delta\boldsymbol{u}=\boldsymbol{0}$, $\delta\varphi=0$ and $\delta\boldsymbol{u}_n=\boldsymbol{0}$ along $\partial\mathcal{M}_d$, so that the second integral in (15) and the last out-of-integral term identically vanish.

This requires the displacement boundary conditions $\boldsymbol{u}-\boldsymbol{u}^{*}=\mathbf{0}$, $\varphi-\varphi^{*}=0$ along $\partial\mathcal{M}_d$ to be satisfied and $\boldsymbol{u}_n-\boldsymbol{u}_n^{*}=\mathbf{0}$ should be assured at each point of irregularity $M_n \in \partial\mathcal{M}_d$, where the starred symbols mean the prescribed quantities.

Then with $(15)_1$ as the second row of (12), the principle of virtual displacements requires the following local Lagrangian equilibrium conditions to be satisfied:

- the equilibrium equations

$$\left(\boldsymbol{T}^{\alpha}+m^{\alpha}\bar{\boldsymbol{n}}\right)|_{\alpha}+\boldsymbol{f}=\mathbf{0} \qquad \text{in } \mathcal{M}, \tag{17}$$

- the natural force static boundary conditions

$$\boldsymbol{P}+m_{\nu}\bar{\boldsymbol{n}}-\boldsymbol{P}^{*}=\mathbf{0}, \quad M-M^{*}=0 \qquad \text{along } \partial\mathcal{M}_f, \tag{18}$$

- the concentrated forces $\boldsymbol{F}_n-\boldsymbol{F}_n^{*}$ applied to each point of irregularity $M_n \in \partial\mathcal{M}_f$.

Particularly simple useful expression for φ was proposed in Pietraszkiewicz and Szwabowicz (1981) as

$$\varphi \equiv n_{\nu}=\bar{\boldsymbol{n}}\cdot\boldsymbol{\nu}=\frac{1}{j}\left(\boldsymbol{u}'\times\boldsymbol{\nu}-\boldsymbol{n}\right)\cdot\boldsymbol{u},_{\nu} \tag{19}$$

following from (8). Varying the two constraints along $\partial\bar{\mathcal{M}}$, one obtains $-\delta\bar{\boldsymbol{n}}\cdot\bar{\boldsymbol{r}}'=\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}'$, $\delta\bar{\boldsymbol{n}}\cdot\bar{\boldsymbol{n}}=0$ from which it follows that $\delta\bar{\boldsymbol{n}}\times\left(\bar{\boldsymbol{r}}'\times\bar{\boldsymbol{n}}\right)=\bar{\boldsymbol{n}}\left(\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}'\right)$. The vector product of this formula by $\boldsymbol{\nu}$ from the left side, after transformations leads to

$$\delta\bar{\boldsymbol{n}}=\frac{1}{e_{\nu}}\left[\left(\bar{\boldsymbol{r}}'\times\bar{\boldsymbol{n}}\right)\delta n_{\nu}+\left(\boldsymbol{\nu}\times\bar{\boldsymbol{n}}\right)\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}'\right], \quad e_{\nu}=\boldsymbol{\nu}\cdot\left(\bar{\boldsymbol{r}}'\times\bar{\boldsymbol{n}}\right), \tag{20}$$

so that in the formula (14) one has

$$\boldsymbol{q}=\frac{1}{e_{\nu}}\left(\bar{\boldsymbol{r}}'\times\bar{\boldsymbol{n}}\right), \quad \boldsymbol{L}=\frac{1}{e_{\nu}}\left(\boldsymbol{\nu}\times\bar{\boldsymbol{n}}\right)\otimes\bar{\boldsymbol{n}}, \tag{21}$$

where $\otimes$ is the tensor product.

Other particular cases of φ suggested in Makowski and Pietraszkiewicz (1989) are: the function $\vartheta_{\nu}=a_{\tau}^{-2}\left(\bar{\boldsymbol{n}}-\boldsymbol{n}\right)\cdot\bar{\boldsymbol{a}}_{\nu}$ proposed by Novozhilov and Shamina (1975) and the angle ω_{τ} of total rotation of the boundary element defined in Pietraszkiewicz (1979) by the relation $2\cos\omega_{\tau}=\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu}+\bar{\boldsymbol{\tau}}\cdot\boldsymbol{\tau}+\bar{\boldsymbol{n}}\cdot\boldsymbol{n}-1$. Corresponding formulas for $\boldsymbol{q},\boldsymbol{L}$ and for the boundary conditions were given in Pietraszkiewicz (1989, 1993).

The Lagrangian vector shell relations derived above have their natural scalar representations in terms of translations u_{α}, w in the known undeformed base $\boldsymbol{a}_{\alpha},\boldsymbol{n}$ of $\mathcal{M}$, displacements $u_{\nu}, u_{\tau}, w, \varphi$ in the known undeformed base $\boldsymbol{\nu},\boldsymbol{\tau},\boldsymbol{n}$ along $\partial\mathcal{M}$ and the surface stress resultants and stress couples $N^{\alpha\beta}, M^{\alpha\beta}$ (Pietraszkiewicz 1984, 1989). These scalar relations are very complex, because they are still valid for unrestricted surface deformation measures $\gamma_{\alpha\beta},\kappa_{\alpha\beta}$ and unrestricted displacements $\boldsymbol{u}$ on $\mathcal{M}$ and $\boldsymbol{u},\varphi$ along $\partial\mathcal{M}$.

Small Elastic Strains

When the strains in the shell space are assumed to be small, i.e. $\max\left(\gamma_{\alpha\beta}, h\kappa_{\alpha\beta}\right) = \eta << 1$, some shell relations derived so far can be consistently simplified. In particular, in $\mathcal{M}$ one has

$$j \simeq 1, \quad \bar{a}^{\alpha\beta} \simeq a^{\alpha\beta} - 2\gamma^{\alpha\beta} \simeq a^{\alpha\beta}, \quad n_\mu \simeq \varphi_\lambda l^\lambda_{\cdot\mu} - \varphi_\mu l^\lambda_{\cdot\lambda}, \quad n \simeq \frac{1}{2}\left(l^\lambda_\lambda l^\mu_\mu - l^\lambda_{\cdot\mu} l^\mu_{\cdot\lambda}\right). \qquad (22)$$

With (22) the surface bendings defined in $(6)_2$ become the third-degree polynomials in u_α, w as well as their surface first and second derivatives.

Along $\partial\mathcal{M}$ one can simplify some relations into

$$j \simeq 1, \quad \bar{\boldsymbol{n}} \simeq \bar{\boldsymbol{r}},_\nu \times \bar{\boldsymbol{r}}', \quad n_\nu \simeq \left(\boldsymbol{u}' \times \boldsymbol{\nu} - \boldsymbol{n}\right) \cdot \boldsymbol{u},_\nu, \qquad (23)$$

so that n_ν becomes the quadratic polynomial in the displacement derivatives.

With (22) the simplified scalar equilibrium equations (17) were explicitly given in Pietraszkiewicz (1989), while with (23) the simplified scalar static boundary conditions were formulated in Pietraszkiewicz (2001).

When the shell is made of an elastic material, the principle (10) requires the existence of the surface strain energy density $\Sigma\left(\gamma_{\alpha\beta}, \kappa_{\alpha\beta}\right)$, per unit area of $\mathcal{M}$, such that $N^{\alpha\beta} = \partial\Sigma / \partial\gamma_{\alpha\beta}$ and $M^{\alpha\beta} = \partial\Sigma / \partial\kappa_{\alpha\beta}$. The explicit expression for Σ depends on the shell material properties, but also on the undeformed shell geometry: its thickness, curvatures of $\mathcal{M}$, the internal structure across the thickness, etc.

In case of a homogeneous isotropic shell undergoing small elastic strains, already Love (1927) used $\Sigma\left(\gamma_{\alpha\beta}, \kappa_{\alpha\beta}\right)$ consisting of the sum of two quadratic functions describing stretching and bending energies of the shell. The error of such approximation was analysed in several papers. In particular, according to Koiter (1960, 1966, 1980) and John (1965) the consistently approximated strain energy density is given indeed by the sum of two quadratic functions

$$\begin{aligned} \Sigma &= \frac{h}{2} H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}\right) + O\left(Eh\eta^2\theta^2\right), \\ H^{\alpha\beta\lambda\mu} &= \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right). \end{aligned} \qquad (24)$$

Here E is the Young modulus and ν is the Poisson ratio of the linearly elastic isotropic material, while the small parameter θ is defined by

$$\theta = \max\left(\frac{h}{b}, \frac{h}{l}, \frac{h}{L}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\right), \qquad (25)$$

where b is the distance from the lateral shell boundary surface, l is the smallest wave length of geometric patterns of $\mathcal{M}$, L is the smallest wave length of deformation patterns on $\mathcal{M}$, and $O(.)$ means “of the order of”. The material tensor $H^{\alpha\beta\lambda\mu}$ corresponds to the plane stress state in the shell in accordance with the assumption 3. indicated in Introduction.

With $(24)_1$, the constitutive equations are defined by

$$N^{\alpha\beta} = \frac{\partial \Sigma}{\partial \gamma_{\alpha\beta}} = \frac{Eh}{1-\nu^2}\left[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^{\kappa}_{\kappa}\right] + O\left(Eh\eta\theta^2\right),$$
$$N^{\alpha\beta} = \frac{\partial \Sigma}{\partial \kappa_{\alpha\beta}} = \frac{Eh^3}{12(1-\nu^2)}\left[(1-\nu)\kappa^{\alpha\beta} + \nu a^{\alpha\beta}\kappa^{\kappa}_{\kappa}\right] + O\left(Eh^2\eta\theta^2\right). \tag{26}$$

Summarizing, the boundary value Lagrangian equilibrium problem of thin isotropic shells undergoing small elastic strains can be formulated in terms of three translations u_α, w of $\mathcal{M}$ as the basic independent field variables. The BVP consists of three scalar equilibrium equations (17) in $\mathcal{M}$, the four natural static (18) and/or four work-conjugate displacement boundary conditions along $\partial\mathcal{M}_f$ or $\partial\mathcal{M}_d$, the concentrated forces $\boldsymbol{F}_n - \boldsymbol{F}_n^*$ applied to each point of irregularity $M_n \in \partial\mathcal{M}_f$, the constitutive equations (26), and the kinematic relations (6) in which the approximate relations (22) and (23) have been used. Unfortunately, these consistently simplified relations are still too complex for engineering applications.

Restricted Rotations

According to the Cauchy theorem, the shell deformation about a point of $\mathcal{M}$ can be exactly decomposed into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation. Since in the previous section the shell relations have been consistently simplified under small elastic strains, several simpler versions of the displacement shell relations can be constructed by imposing consistent restrictions upon the rotations of the shell material elements (Pietraszkiewicz 1977, 1980, 1984).

A finite rotation may be described by an angle of rotation ω about an axis of rotation fixed in space by a unit vector $\boldsymbol{e}$. In mathematics the rotation is usually defined by the 2nd-order tensor $\boldsymbol{R}(\omega, \boldsymbol{e})$ such that $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$, $\det \boldsymbol{R} = +1$. Alternatively, for $|\omega| < \pi/2$ the rotation can be uniquely described by some finite rotation vector such as $\boldsymbol{\Omega} = \boldsymbol{e}\sin\omega$ or $\boldsymbol{\theta} = \boldsymbol{e}\,2\tan\omega/2$. The magnitude of the rotation can be classified in terms of the small parameter θ defined in (25) as follows: a) $\omega \le O(\theta^2)$ - the small rotation, b) $\omega = O(\theta)$ - the moderate rotation, c) $\omega = O(\theta\sqrt{\theta})$ - the large rotation, and d) $\omega = O(1)$ - the finite rotation. However, the shell structures are usually quite rigid for in-surface deformation being flexible for out-of-surface deformation. To account this property, one can associate the names "small, moderate, large, finite" rotation with the particular components $\Omega_\alpha = \boldsymbol{\Omega}\cdot\boldsymbol{a}_\alpha$ and $\Omega = \boldsymbol{\Omega}\cdot\boldsymbol{n}$ of $\boldsymbol{\Omega}$.

In the geometrically non-linear theory of thin shells $\boldsymbol{\Omega}$ is expressed through translations of $\mathcal{M}$ by the consistent reduction of the exact formula (Pietraszkiewicz 1977, f. (3.7.17) or 1984, f. (2.3.11)):

$$\boldsymbol{\Omega} \simeq \varepsilon^{\beta\alpha}\left[\varphi_\alpha\left(1 + \frac{1}{2}\theta^{\kappa}_{\kappa}\right) - \frac{1}{2}\varphi^{\lambda}\left(\theta_{\lambda\alpha} - \omega_{\lambda\alpha}\right)\right]\boldsymbol{a}_\beta + \phi\boldsymbol{n}, \tag{27}$$

$$\theta_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) - b_{\alpha\beta} w , \quad \omega_{\alpha\beta} = \frac{1}{2}\left(u_{\beta|\alpha} - u_{\alpha|\beta}\right) ,$$
$$\phi = \frac{1}{2}\varepsilon^{\alpha\beta} u_{\beta|\alpha} , \quad \boldsymbol{\phi} = \varepsilon^{\beta\alpha}\varphi_\alpha + \phi\boldsymbol{n} . \tag{28}$$

Here $\theta_{\alpha\beta}$ are components of the linearized strains while φ_α, ϕ describe the linearized rotation vector $\boldsymbol{\phi}$.

For any restriction imposed on $\boldsymbol{\Omega}$ the estimates for φ_α, ϕ follow from (27) and those for $\theta_{\alpha\beta}$ follow from solving $(6)_1$ with $\gamma_{\alpha\beta} = O(\eta)$. Then the consistently simplified expressions for $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ can be established taking account of accuracy of Σ in $(24)_1$. In such estimation procedure covariant surface derivatives are estimated dividing their maximal value by the large parameter

$$\lambda = \frac{h}{\theta} = \min\left(b, l, L, \sqrt{hR}, \frac{1}{\sqrt{\eta}} \right) . \tag{29}$$

Within *small rotations* $\varphi_\alpha = O(\theta^2)$, $\omega_{\alpha\beta} = O(\theta^2)$, $\theta_{\alpha\beta} = O(\theta^2)$ and the shell deformation measures are consistently approximated by $\gamma_{\alpha\beta} = \theta_{\alpha\beta} + O(\eta\theta^2)$, $\kappa_{\alpha\beta} = -1/2(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) + O(\eta\theta/\lambda)$. These characterize the linear bending theory of thin elastic shells treated in many books and papers, see for example (Koiter 1961).

Within *moderate rotations* $\varphi_\alpha = O(\theta)$, $\omega_{\alpha\beta} = O(\theta)$, $\theta_{\alpha\beta} = O(\theta^2)$ and the consistently reduced shell relations are

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + \frac{1}{2}\omega^{\lambda}_{\cdot\alpha}\omega_{\lambda\beta} - \frac{1}{2}\left(\theta^{\lambda}_{\alpha}\omega_{\lambda\beta} + \theta^{\lambda}_{\beta}\omega_{\lambda\alpha}\right) + O(\eta\theta^2) ,$$
$$\kappa_{\alpha\beta} = -\frac{1}{2}\left(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} - b^{\lambda}_{\alpha}\omega_{\lambda\beta} - b^{\lambda}_{\beta}\omega_{\lambda\alpha}\right) + O\left(\frac{\eta\theta}{\lambda}\right) , \tag{30}$$

$$\boldsymbol{T}^{\alpha} = \left\{ N^{\alpha\lambda} - \frac{1}{2}\left(b^{\alpha}_{\beta}M^{\lambda\beta} + b^{\lambda}_{\beta}M^{\alpha\beta}\right) + \frac{1}{2}N^{\kappa}_{\kappa}\omega^{\alpha\lambda} - \frac{1}{2}\left(N^{\alpha}_{\beta}\omega^{\lambda\beta} + N^{\lambda}_{\beta}\omega^{\alpha\beta}\right) \right.$$
$$\left. + \frac{1}{2}\left(N^{\alpha}_{\beta}\theta^{\lambda\beta} - N^{\lambda}_{\beta}\theta^{\alpha\beta}\right) \right\}\boldsymbol{a}_\lambda + \left(N^{\alpha\beta}\varphi_\beta + M^{\beta\alpha}|_\beta\right)\boldsymbol{n} , \tag{31}$$

$$\bar{\boldsymbol{n}} = -\varphi_\alpha \boldsymbol{a}^\alpha + \boldsymbol{n} , \quad n_\nu = -\varphi_\nu ,$$
$$\boldsymbol{F} = M_{\nu\tau}\boldsymbol{n} , \quad M = M_{\nu\nu} , \quad \boldsymbol{F}^* = m^*_\tau \boldsymbol{n} , \quad M^* = m^*_\nu . \tag{32}$$

If additionally the rotation about normal Ω is restricted to be small then also $\omega_{\alpha\beta} = O(\theta^2)$. For such *moderate/small rotation* theory of thin elastic shells the relations (30) and (31) can be considerably simplified to

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + O(\eta\theta^2) , \quad \kappa_{\alpha\beta} = -\frac{1}{2}\left(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}\right) + O\left(\frac{\eta\theta}{\lambda}\right) ,$$
$$\boldsymbol{T}^{\alpha} = N^{\alpha\lambda}\boldsymbol{a}_\lambda + \left(N^{\alpha\beta}\varphi_\beta + M^{\beta\alpha}|_\beta\right)\boldsymbol{n} . \tag{33}$$

The non-linear relations (30) and (31) (or (33)) with (32) and (26), when introduced into the equilibrium conditions (17) and (18), and the surface couples $\boldsymbol{m}$ conventionally omitted, describe the boundary value problem in displacements of the Lagrangian geometrically non-linear theory of thin elastic shells undergoing moderate (or moderate/small) rotations. This version of shell equations contains as special cases a number of simpler versions of non-linear shell equations proposed in the literature. A detailed review of those simpler versions was given in Schmidt and Pietraszkiewicz (1981), where also a set of sixteen basic free functionals and several functionals with subsidiary conditions were constructed for conservative dead-load type surface and boundary loadings.

The simplest case of the moderate/small rotation theory is the non-linear theory of shallow shells proposed by Mushtari (1939) and Marguerre (1939). In this case one additionally assumes that the tangential surface forces f^{α} are also small and can be ignored. As a result, the tangential translations can be expected to be one order smaller than the normal ones, $u_{\alpha} = w \cdot O(\theta)$, so that $\varphi_{\alpha} = w,_{\alpha} [1 + O(\theta^2)]$ and the surface deformation measures become extremely simple

$$\gamma_{\alpha\beta} \simeq \theta_{\alpha\beta} + \frac{1}{2} w,_{\alpha} w,_{\beta} , \quad \kappa_{\alpha\beta} \simeq -w_{|\alpha\beta} . \tag{34}$$

In the tangential equilibrium equations the terms $-b^{\lambda}_{\alpha} T^{\alpha}$ are small as compared with $N^{\alpha\lambda}|_{\alpha}$ and can be omitted. As a result, the equilibrium equations become extremely simple as well,

$$N^{\alpha\beta}|_{\alpha} = 0 , \quad M^{\alpha\beta}|_{\alpha\beta} + \left(b_{\alpha\beta} + w_{|\alpha\beta}\right) N^{\alpha\beta} + f = 0 . \tag{35}$$

The equilibrium equations (35) together with the constitutive relations (26), the kinematic relations (34) and corresponding boundary conditions form the boundary value problem of the non-linear theory of shallow elastic shells expressed in terms of translations of $\mathcal{M}$ as the independent variables.

It can be proved that the approximate expression $(34)_2$ for $\kappa_{\alpha\beta}$ satisfies approximately two tangential compatibility conditions, which suggests that within this approximation the order of covariant differentiation is unimportant. This allows one to approximately satisfy $(35)_1$ by $N^{\alpha\beta} = \varepsilon^{\alpha\lambda} \varepsilon^{\beta\mu} \Psi_{|\lambda\mu}$, where Ψ is the stress function. Then $(35)_2$ and the third compatibility condition for $\gamma_{\alpha\beta}, \kappa_{\alpha\beta}$ leads to the set of two equations

$$\begin{aligned} &\frac{Eh^3}{12(1-\nu^2)} w|^{\alpha\beta}_{\alpha\beta} - \varepsilon^{\alpha\lambda} \varepsilon_{\beta\mu} \left(b^{\beta}_{\alpha} + w|^{\beta}_{\alpha}\right) \Psi|^{\mu}_{\lambda} - f = 0 , \\ &\frac{1}{Eh} \Psi|^{\alpha\beta}_{\alpha\beta} + \varepsilon^{\alpha\lambda} \varepsilon_{\beta\mu} \left(b^{\beta}_{\alpha} + \frac{1}{2} w|^{\beta}_{\alpha}\right) w|^{\mu}_{\lambda} = 0 . \end{aligned} \tag{36}$$

These two equations usually written in the orthogonal lines of principal curvatures of $\mathcal{M}$, together with corresponding boundary conditions expressed in w, Ψ, are given in many books and papers, for example Mushtari and Galimov (1961), and Brush and Almroth (1975), where also many numerical examples are presented.

The consistent relations of the *large/small rotation* Lagrangian non-linear theory of shells have been presented in detail in Pietraszkiewicz (1984, 1989). These relations are more involved than those of the moderate rotation theory. As a result, they have been used to solve some engineering shell problems in only limited number of papers.

Cross References

Surface Geometry, Elements
Thin Elastic Shells, Linear Theory
Junctions in Irregular Shell Structures
Elastic Shells, Resultant Non-linear Theory

References

Brush DO, Almroth BO (1975) Buckling of bars, plates and shells. McGraw-Hill, New York

Galimov KZ (1951) On the general theory of plates and shells with finite displacements and deformations (In Russian). Prikl Matem Mekh 15(6):723-742

John F (1965) Estimates for the derivatives of the stresses in a thin shell and interior shell equations. Comm Pure Appl Math 18:235-267.

Koiter WT (1960) A consistent first approximation in the general theory of thin elastic shells. Proc IUTAM Symp Theory of Shells. North-Holland, Amsterdam, pp 12-33

Koiter WT (1961) A systematic simplification of the general equations in the linear theory of thin shells. Proc Koninkl Nederl Acad Wetensch Ser B 64:612-619.

Koiter WT (1966) On the nonlinear theory of thin elastic shells. Proc Koninkl Nederl Acad Wetensch Ser B 69(1):1-54

Koiter WT (1980) The intrinsic equations of shell theory with some applications. In: Nemat-Nasser (ed), Mechanics Today 5. Pergamon Press, Oxford, p. 139-154

Love AEH (1927) A treatise on the mathematical theory of elasticity, 4^{th} edn. University Press, Cambridge

Makowski J, Pietraszkiewicz W (1989) Work-conjugate boundary conditions in the nonlinear theory of thin shells. Trans ASME, J Appl Mech 56(2):395-402

Marguerre K (1939) Zur Theorie der gekrümmten Platte grosser Formänderung. Proc 5^{th} Int Congr Appl Mech. Cambridge, New York, pp 93-101

Mushtari KM (1939) Some generalizations of the theory of thin shells with applications to the solution of stability problems of elastic equilibrium (in Russian) Prikl Matem Mekh 2(4):439-456

Mushtari KM, Galimov KZ (1961) Non-linear theory of thin elastic shells. The Israel Program for Sci Transl, Jerusalem

Novozhilov VV, Shamina VA (1975) On kinematic boundary conditions in non-linear problems of theory of elasticity (in Russian). Mekh Tverdogo Tela 5:63-71

Pietraszkiewicz W (1977) An introduction to the non-linear theory of shells. Ruhr-Universität, Mitt Inst für Mech 10, Bochum

Pietraszkiewicz W (1980) Finite rotations in the nonlinear theory of thin shells. In: Olszak W (ed) Thin Shell Theory: New Trends and Applications. CISM Courses and Lectures No 240. Springer-Verlag, Wien-New York, p. 154-208.

Pietraszkiewicz W (1984) Lagrangian description and incremental formulation in the non-linear theory of thin shells. Int J Non-Linear Mech 19(2):115-140.

Pietraszkiewicz W (1989) Geometrically nonlinear theories of thin elastic shells. Advances in Mechanics 12(1):51-130.

Pietraszkiewicz W (1993) Work-conjugate boundary conditions associated with the total rotation angle of the shell boundary. Trans ASME, J Appl Mech 60(3):785-786.

Pietraszkiewicz W (2001) Non-linear theories of shells. In: Woźniak C (ed) Mechanics of Elastic Plates and Shells (in Polish). Wyd Nauk PWN, Warszawa, p.424-497.

Pietraszkiewicz W (2017) Surface geometry, Elements. In: Altenbach H, Őchsner A (eds) Encyclopedia of Continuum mechanics, Section: Shells. Springer-Verlag, Berlin (in print).

Pietraszkiewicz W (2017a) Thin elastic shells, Linear theory. In: Altenbach H, Őchsner A (eds) Encyclopedia of Continuum mechanics, Section: Shells. Springer-Verlag, Berlin (in print).

Pietraszkiewicz W, Szwabowicz ML (1981) Entirely Lagrangian nonlinear theory of thin shells. Arch Mech 33(2):273-288.

Schmidt R, Pietraszkiewicz W (1981) Variational principles in the geometrically non-linear theory of shells undergoing moderate rotations. Ing-Archiv 50:187-201.