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Thin Elastic Shells, Linear Theory

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Synonyms

Kirchhoff-Love shell model; Classical shell theory

Definition

The linear theory of thin elastic shells is an approximate two-dimensional case of three-dimensional linear theory of elasticity. Here the shell thickness is supposed to be much smaller than the smallest radius of curvature of the shell middle surface. As a result, the shell deformation can approximately be described only by stretching and bending of its middle surface.

Introduction

The linear theory of thin elastic shells belongs to classical special two-dimensional models within linear elasticity. It was originated by Love (1888), developed subsequently in thousands of papers and summarized in dozens of monographs. Among the most important books in the field one might mention Love (1927), Goldenveizer (1961), Novozhilov (1964), Başar and Krätzig (1985), and Novozhilov et al. (1991).

Initially, the linear thin shell relations were developed in orthogonal coordinates coinciding with lines of principal curvatures on the shell middle surface. However, such a shape-dependent description was found to be too complex and inefficient due to a large variety of shell shapes appearing in technology and everyday life. Lurie (1940) proposed to describe the linear shell relations in the invariant tensor notation. These relations were valid for any geometry of the shell midsurface. This compact description was then applied in many papers and books, for example by Green and Zerna (1954), Koiter (1960), Naghdi (1963), Chernykh (1964), Flügge (1972), Başar and Krätzig (1985, 2001), and Ciarlet (2000).

In this note basic relations of the linear theory of thin isotropic elastic shells are briefly derived and discussed. The formulation is based on the following simplifying assumptions:

1. The material elements, which are normal to the shell middle surface in the undeformed placement, remain normal to the deformed shell midsurface and do not change their lengths. This assumption allows one to approximately describe the three-dimensional shell deformation only by stretching and bending of its middle surface.
2. The state of elastic stress in the shell space is approximately plane. This means that the effects of transverse shear stresses and of normal stresses, acting on surfaces parallel to the middle surface, may be neglected in the elastic strain energy density.

To be concise, these assumptions are used here to derive the approximate two-dimensional equilibrium equations and boundary conditions from the postulated principle of virtual displacements for the shell midsurface. In deriving the basic shell relations the tensor notation for description of surface geometry is applied according to Pietraszkiewicz (2018).

Geometry and Small Deformation of a Thin Shell

Let $\mathcal{P}$ be the region of three-dimensional Euclidean point space $\mathcal{E}$ occupied by the shell in the undeformed placement. The position vector of any point $\mathrm{P} \in \mathcal{P}$ relative to a reference point $\mathrm{O} \in \mathcal{E}$ can be given by

$$\mathbf{p}(\theta^\alpha, \xi) = \mathrm{p}^i(\theta^\alpha)\mathbf{i}_i = \boldsymbol{r}(\theta^\alpha) + \xi \boldsymbol{n}(\theta^\alpha)\,, \tag{1}$$

where $\mathbf{i}_i\,, i = 1,2,3,$ are three orthonormal vectors of a reference frame, $\theta^\alpha, \alpha = 1,2,$ are the curvilinear surface coordinates, $-h/2 \le \xi \le +h/2$ is the distance from the shell middle surface $\mathcal{M}$ defined by the position vector $\boldsymbol{r}(\theta^\alpha)$, $\boldsymbol{n}(\theta^\alpha)$ is the unit normal vector orienting $\mathcal{M}$, and h is the shell thickness. In thin shell theory it is understood that h be constant and small as compared with the smallest radius of curvature R of $\mathcal{M}$, i.e. $h/R << 1,$ and with linear dimensions of $\mathcal{P}$.

Geometry of the base surface $\mathcal{M}$ is described by the following fields (Pietraszkiewicz 2018): the natural base vectors $\boldsymbol{a}_\alpha = \partial \boldsymbol{r} / \partial \theta^\alpha \equiv \boldsymbol{r},_\alpha$, the covariant components $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ of the surface metric tensor, the dual (contravariant) base vectors $\boldsymbol{a}^\beta$ such that $\boldsymbol{a}_\alpha \cdot \boldsymbol{a}^\beta = \delta_\alpha^\beta$, where $\delta_1^1 = \delta_2^2 = 1, \delta_2^1 = \delta_1^2 = 0$, the unit normal vector $\boldsymbol{n} = \boldsymbol{a}_1 \times \boldsymbol{a}_2 / |\boldsymbol{a}_1 \times \boldsymbol{a}_2|$, and the covariant components $b_{\alpha\beta} = -\boldsymbol{a}_\alpha \cdot \boldsymbol{n},_\beta = -\boldsymbol{n},_\alpha \cdot \boldsymbol{a}_\beta$ of the surface curvature tensor. The contravariant metric components $a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$ are used to raise indices of surface vectors and tensors, for example $\boldsymbol{a}^\beta = a^{\beta\alpha}\boldsymbol{a}_\alpha$, $b_\beta^\alpha = a^{\alpha\lambda} b_{\lambda\beta}$, etc.

The boundary contour $\partial\mathcal{M}$ of $\mathcal{M}$ consists of a finite set of piecewise smooth curves $\boldsymbol{r}(s) = \boldsymbol{r}[\theta^\alpha(s)]$, where s is the arc length along $\partial\mathcal{M}$. In each regular point $M \in \partial\mathcal{M}$ two unit vectors are defined: the tangent $\boldsymbol{\tau} = d\boldsymbol{r}/ds \equiv \boldsymbol{r}' = \boldsymbol{a}_\alpha \tau^\alpha$ and the outward normal $\boldsymbol{\nu} = \boldsymbol{\tau} \times \boldsymbol{n} = \boldsymbol{a}_\alpha \nu^\alpha \equiv \boldsymbol{r},_\nu$, $\nu^\alpha = \varepsilon^{\alpha\beta}\tau_\beta$, where $\boldsymbol{r},_\nu$ means the outward normal derivative of $\boldsymbol{r}$ at $\partial\mathcal{M}$ and $\varepsilon^{\alpha\beta}$ are contravariant components of the surface alternation tensor.

The shell midsurface in the deformed placement $\bar{\mathcal{M}}$ is described by the position vector $\bar{\boldsymbol{r}}(\theta^\alpha) = \boldsymbol{r}(\theta^\alpha) + \boldsymbol{u}(\theta^\alpha)$, where θ^α are the surface curvilinear convective coordinates and $\boldsymbol{u} = u_\alpha \boldsymbol{a}^\alpha + w\boldsymbol{n}$ is the surface translation field. The geometric quantities describing $\bar{\mathcal{M}}$ are analogous to those of $\mathcal{M}$, only now they are marked by the overbar, for example $\bar{\boldsymbol{a}}_\alpha, \bar{\boldsymbol{n}}, \bar{a}_{\alpha\beta}, \bar{b}_{\alpha\beta},$ etc. The barred quantities can be expressed through analogous unbarred ones and the vector $\boldsymbol{u}$. In the linear theory of thin shells these relations are approximated by linear functions of $\boldsymbol{u}$. In particular, one has (Koiter 1960; Pietraszkiewicz 1980)

$$\begin{gathered}\bar{\boldsymbol{a}}_\alpha = \boldsymbol{a}_\alpha + \boldsymbol{u},_\alpha = \left(a_{\lambda\alpha} + \theta_{\lambda\alpha} - \omega_{\lambda\alpha}\right)\boldsymbol{a}^\lambda + \varphi_\alpha \boldsymbol{n}\,, \quad \bar{\boldsymbol{n}} = \boldsymbol{n} - \varphi_\alpha \boldsymbol{a}^\alpha\,, \\ \theta_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) - b_{\alpha\beta} w\,, \quad \omega_{\alpha\beta} = \frac{1}{2}\left(u_{\beta|\alpha} - u_{\alpha|\beta}\right), \quad \varphi_\alpha = w,_\alpha + b_\alpha^\lambda u_\lambda\,,\end{gathered} \tag{2}$$

$$\bar{a}_{\alpha\beta} = \bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{a}}_\beta = a_{\alpha\beta} + 2\theta_{\alpha\beta} \,,$$
$$\bar{b}_{\alpha\beta} = -\bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{n}},_\beta = b_{\alpha\beta} + \varphi_{\alpha|\beta} + b^\lambda_\beta \left(\theta_{\lambda\alpha} - \omega_{\lambda\alpha}\right), \tag{3}$$
$$\bar{b}_{\alpha\beta} = -\bar{\boldsymbol{a}}_\beta \cdot \bar{\boldsymbol{n}},_\alpha = b_{\alpha\beta} + \varphi_{\beta|\alpha} + b^\lambda_\alpha \left(\theta_{\lambda\beta} - \omega_{\lambda\beta}\right),$$

where $(.)_{|\alpha}$ denotes the covariant surface differentiation of $(.)$.

In the linear theory of thin shells the surface deformation measures are linear functions of the surface translations and their surface derivatives, and are defined by

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(\bar{a}_{\alpha\beta} - a_{\alpha\beta}\right) = \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha}\right) - b_{\alpha\beta} w \,,$$
$$\kappa_{\alpha\beta} = -\left(\bar{b}_{\alpha\beta} - b_{\alpha\beta}\right) = -w_{|\alpha\beta} - b^\lambda_\alpha u_{\lambda|\beta} - b^\lambda_\beta u_{\lambda|\alpha} - b^\lambda_{\alpha|\beta} u_\lambda + b^\lambda_\alpha b_{\lambda\beta} w \,. \tag{4}$$

The components $\gamma_{\alpha\beta}$ are the linearized surface strains while $\kappa_{\alpha\beta}$ are the linearized surface changes of curvatures (briefly bendings). Both surface deformation measures are symmetric: $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}$, $\kappa_{\alpha\beta} = \kappa_{\beta\alpha}$. Please note that $\kappa_{\alpha\beta}$ are given through the surface translations as well as their first and second surface derivatives.

Equilibrium Conditions

Let the shell base surface $\mathcal{M}$ be loaded by the distributed force field $\boldsymbol{f} = f^\alpha \boldsymbol{a}_\alpha + f\boldsymbol{n}$ per unit area of $\mathcal{M}$, as well as by the boundary force $\boldsymbol{n}^* = n^*_\nu \boldsymbol{\nu} + n^*_\tau \boldsymbol{\tau} + n^* \boldsymbol{n}$ and the boundary couple $\boldsymbol{m}^* = \boldsymbol{n} \times \left(m^*_\nu \boldsymbol{\nu} + m^*_\tau \boldsymbol{\tau}\right)$, both per unit length of the boundary contour $\partial\mathcal{M}_f$. If the shell is to be in equilibrium then, within the simplifying assumptions of the linear thin shell model, for a virtual displacement field $\delta\boldsymbol{u} = \delta u_\alpha \boldsymbol{a}^\alpha + \delta w \boldsymbol{n}$ subjected to kinematic constraints the following principle of virtual displacements should be satisfied:

$$\iint_{\mathcal{M}} \left(N^{\alpha\beta} \delta\gamma_{\alpha\beta} + M^{\alpha\beta} \delta\kappa_{\alpha\beta}\right) da = \iint_{\mathcal{M}} \boldsymbol{f} \cdot \delta\boldsymbol{u} \, da + \int_{\partial\mathcal{M}_f} \left(\boldsymbol{n}^* \cdot \delta\boldsymbol{u} + \boldsymbol{m}^* \cdot \delta\boldsymbol{\omega}_\tau\right) ds \,. \tag{5}$$

Here $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are symmetric components of the internal surface stress and couple resultants, $\partial\mathcal{M}_f$ is that part of $\partial\mathcal{M}$ along which the forces $\boldsymbol{n}^*$ and couples $\boldsymbol{m}^*$ are prescribed, and $\boldsymbol{\omega}_\tau$ is the linearized rotation vector of the shell lateral boundary element.

The first surface integral of (5) indicates the internal virtual work G_{int} performed by $N^{\alpha\beta}$, $M^{\alpha\beta}$ on the respective virtual strains and bendings $\delta\gamma_{\alpha\beta}, \delta\kappa_{\alpha\beta}$ given according to (2) - (4) by

$$\delta\gamma_{\alpha\beta} = \frac{1}{2}\left(\delta u_{\alpha|\beta} + \delta u_{\beta|\alpha}\right) - b_{\alpha\beta} \delta w \,,$$
$$\delta\kappa_{\alpha\beta} = -\delta w_{|\alpha\beta} - b^\lambda_\alpha \delta u_{\lambda|\beta} - b^\lambda_\beta \delta u_{\lambda|\alpha} - b^\lambda_{\alpha|\beta} \delta u_\lambda + b^\lambda_\alpha b_{\lambda\beta} \delta w \,. \tag{6}$$

The last two integrals of (5) indicate the external virtual work G_{ext} performed by $\boldsymbol{f}$ on $\delta\boldsymbol{u}$ within $\mathcal{M}$ as well as by $\boldsymbol{n}^*$ and $\boldsymbol{m}^*$ on the corresponding $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\omega}_\tau$ along $\partial\mathcal{M}_f$, respectively. The virtual fields should satisfy the kinematic constraints, that is $\delta\boldsymbol{u} = \boldsymbol{0}$ and $\delta\boldsymbol{\omega}_\tau = \boldsymbol{0}$ along $\partial\mathcal{M}_d$, where $\partial\mathcal{M} = \partial\mathcal{M}_f \cup \partial\mathcal{M}_d$.

With the help of some tensor identities, the G_{int} in (5) with (6) can be transformed into

$$G_{\text{int}} = -\iint_{\mathcal{M}} \Big\{ \Big[\Big(N^{\alpha\beta} - b^{\beta}_{\lambda} M^{\lambda\alpha} \Big)|_{\alpha} - b^{\beta}_{\lambda} M^{\lambda\alpha}|_{\alpha} \Big] \delta u_{\beta} + \Big[M^{\alpha\beta}|_{\alpha\beta} + b_{\alpha\beta} \Big(N^{\alpha\beta} - b^{\alpha}_{\lambda} M^{\lambda\beta} \Big) \Big] \delta w$$
$$- \Big[\Big(N^{\alpha\beta} - b^{\beta}_{\lambda} M^{\lambda\alpha} \Big) \delta u_{\beta} - M^{\alpha\beta} \Big(\delta w_{,\beta} + b^{\lambda}_{\beta} \delta u_{\lambda} \Big) + M^{\alpha\beta}|_{\alpha\beta}\, \delta w \Big]\Big|_{\alpha} \Big\} da \,. \tag{7}$$

Introducing the vector

$$\boldsymbol{n}^{\alpha} = \Big(N^{\alpha\beta} - b^{\beta}_{\lambda} M^{\lambda\alpha} \Big) \boldsymbol{a}_{\beta} + M^{\alpha\beta}|_{\beta}\, \boldsymbol{n} \,, \tag{8}$$

and applying the surface divergence theorem to the last term of (7) in brackets, the relation (7) can be written in the compact form

$$G_{\text{int}} = -\iint_{\mathcal{M}} \boldsymbol{n}^{\alpha}|_{\alpha} \cdot \delta \boldsymbol{u}\, da + \int_{\partial \mathcal{M}_f} \Big(\boldsymbol{n}^{\alpha} \cdot \delta \boldsymbol{u} - M^{\alpha\beta} \delta \varphi_{\beta} \Big) \nu_{\alpha}\, ds \,, \tag{9}$$

where $\delta\varphi_{\beta}$ are tangential components of variation of the linearized rotation vector of $\mathcal{M}$,

$$\boldsymbol{\phi} = \varepsilon^{\beta\alpha} \left(\varphi_{\alpha} \boldsymbol{a}_{\beta} + \frac{1}{2} u_{\alpha|\beta} \boldsymbol{n} \right). \tag{10}$$

Along $\partial\mathcal{M}$ the virtual translations and rotations can be expanded into physical components

$$\delta \boldsymbol{u} = \delta u_{\nu} \boldsymbol{\nu} + \delta u_{\tau} \boldsymbol{\tau} + \delta w \boldsymbol{n} \,, \quad \delta\varphi_{\beta} = \delta\varphi_{\nu} \nu_{\beta} + \delta\varphi_{\tau} \tau_{\beta} \,,$$
$$\delta\varphi_{\tau} = \delta\varphi_{\beta} \tau^{\beta} = \frac{d}{ds}(\delta w) + \sigma_{\tau} \delta u_{\tau} - \tau_{\tau} \delta u_{\nu} \,, \tag{11}$$
$$\delta u_{\nu} = \delta u_{\beta} \nu^{\beta} \,, \quad \delta u_{\tau} = \delta u_{\beta} \tau^{\beta} \,, \quad \sigma_{\tau} = b_{\alpha\beta} \tau^{\alpha} \tau^{\beta} \,, \quad \tau_{\tau} = -b_{\alpha\beta} \nu^{\alpha} \tau^{\beta} \,.$$

The first term in $(11)_2$ allows one to integrate by parts the last expressions in the line integral (9) leading to

$$-\int_{\partial\mathcal{M}} \left[M_{\nu\nu} \delta\varphi_{\nu} + M_{\nu\tau} \left(\frac{d}{ds}(\delta w) + \sigma_{\tau} \delta u_{\tau} - \tau_{\tau} \delta u_{\nu} \right) \right] ds$$
$$= \int_{\partial\mathcal{M}} \left[\frac{d}{ds} \big(M_{\nu\tau} \boldsymbol{n} \big) \cdot \delta \boldsymbol{u} - M_{\nu\nu} \delta\varphi_{\nu} \right] ds \tag{12}$$
$$+ \sum_{n=1}^{N} \big[M_{\nu\tau}(s_n + 0) - M_{\nu\tau}(s_n - 0) \big] \delta w(s_n) \,.$$

The linearized rotation vector $\boldsymbol{\omega}_{\tau}$ of the shell lateral boundary element is related to the linearized rotation vector $\boldsymbol{\phi}$ of $\mathcal{M}$ by $\boldsymbol{\omega}_{\tau} = \boldsymbol{\phi} - \gamma_{\nu\tau} \boldsymbol{n}$ (Chernykh 1964; Pietraszkiewicz 1980). But within the assumptions of the linear thin shell theory $\boldsymbol{m}^{*}$ does not have the normal component, so it is always $\boldsymbol{m}^{*} \cdot \boldsymbol{n} \equiv 0$. As a result, $\boldsymbol{m}^{*} \cdot \delta\boldsymbol{\omega}_{\tau} = \boldsymbol{m}^{*} \cdot \delta\boldsymbol{\phi}$ and the last term in (5) can be transformed similarly as in (12) leading to

$$\int_{\partial\mathcal{M}_f} \boldsymbol{m}^{*} \cdot \delta\boldsymbol{\omega}_{\tau}\, ds = \int_{\partial\mathcal{M}_f} \left[\frac{d}{ds} \Big(m^{*}_{\tau} \boldsymbol{n} \Big) \cdot \delta \boldsymbol{u} - m^{*}_{\nu} \delta\varphi_{\nu} \right] ds$$
$$+ \sum_{n=1}^{N} \Big[m^{*}_{\tau}(s_n + 0) - m^{*}_{\tau}(s_n - 0) \Big] \delta w(s_n) \,. \tag{13}$$

Summarizing, the principle of virtual displacements (5) with (6) - (13) requires the following local relations to be satisfied:

- The equilibrium equations

$$\boldsymbol{n}^{\alpha}|_{\alpha} + \boldsymbol{f} = \boldsymbol{0} \quad \text{in}\ \mathcal{M} \,. \tag{14}$$

- The force static boundary conditions

$$\boldsymbol{n}^{\alpha}\nu_{\alpha}+\frac{d}{ds}\left(M_{\nu\tau}\boldsymbol{n}\right)=\boldsymbol{n}^{*}+\frac{d}{ds}\left(m_{\tau}^{*}\boldsymbol{n}\right),\quad M_{\nu\nu}=m_{\nu}^{*}\quad \text{along } \partial\mathcal{M}_{f}\ . \tag{15}$$

- The concentrated forces applied in each singular point $M_n\in\partial\mathcal{M}_f$,

$$\left\{\left[M_{\nu\tau}(s_n+0)-m_{\tau}^{*}(s_n+0)\right]-\left[M_{\nu\tau}(s_n-0)-m_{\tau}^{*}(s_n-0)\right]\right\}\boldsymbol{n}(s_n)\ . \tag{16}$$

- The corresponding work-conjugate displacement boundary conditions are

$$\boldsymbol{u}=\boldsymbol{u}^{*}\ ,\quad \varphi_{\nu}=\varphi_{\nu}^{*}\quad \text{along } \partial\mathcal{M}_{d}\ . \tag{17}$$

In components some of these relations are:

- The equilibrium equations in $\mathcal{M}$,

$$T^{\alpha\beta}|_{\alpha}-b_{\lambda}^{\beta}Q^{\lambda}+f^{\beta}=0\ ,\quad M^{\alpha\beta}|_{\alpha\beta}+b_{\alpha\beta}T^{\alpha\beta}+f=0\ , \tag{18}$$

where the following abbreviations have been used:

$$T^{\alpha\beta}=N^{\alpha\beta}-b_{\lambda}^{\beta}M^{\lambda\alpha}\ ,\quad Q^{\lambda}=M^{\alpha\lambda}\,|_{\alpha}\ . \tag{19}$$

- The force static boundary conditions along $\partial\mathcal{M}_f$,

$$\begin{aligned}
&T_{\nu\nu}+\tau_{\tau}M_{\nu\tau}=n_{\nu}^{*}+\tau_{\tau}m_{\tau}^{*}\quad \text{in direction of } \boldsymbol{\nu},\\
&T_{\nu\tau}-\sigma_{\tau}M_{\nu\tau}=n_{\tau}^{*}-\sigma_{\tau}m_{\tau}^{*}\quad \text{in direction of } \boldsymbol{\tau},\\
&Q_{\nu}+\frac{d}{ds}M_{\nu\tau}=n^{*}+\frac{d}{ds}m_{\tau}^{*}\quad \text{in direction of } \boldsymbol{n},
\end{aligned} \tag{20}$$

where the following abbreviations of the physical components have been used:

$$T_{\nu\nu}=T^{\alpha\beta}\nu_{\alpha}\nu_{\beta}\ ,\quad T_{\nu\tau}=T^{\alpha\beta}\nu_{\alpha}\tau_{\beta}\ ,\quad M_{\nu\nu}=M^{\alpha\beta}\nu_{\alpha}\nu_{\beta}\ ,\quad M_{\nu\tau}=M^{\alpha\beta}\nu_{\alpha}\tau_{\beta}\ ,\quad Q_{\nu}=Q^{\alpha}\nu_{\alpha}\ . \tag{21}$$

Compatibility Conditions

Six components $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are expressed by only three components of $\boldsymbol{u}$ on $\mathcal{M}$. Thus, the surface deformation measures have to satisfy three compatibility conditions.

Two smooth and differentiable vector displacement fields on the regular shell midsurface $\mathcal{M}$ satisfy the obvious identities $\boldsymbol{u},_{12}=\boldsymbol{u},_{21}$ and $\boldsymbol{\phi},_{12}=\boldsymbol{\phi},_{21}$, which can equivalently be written as

$$\varepsilon^{\alpha\beta}\boldsymbol{u}_{|\alpha\beta}=\boldsymbol{0}\ ,\quad \varepsilon^{\alpha\beta}\boldsymbol{\phi}_{|\alpha\beta}=\boldsymbol{0}\ . \tag{22}$$

Differentiation of $\boldsymbol{u}$ with the use of (10) leads to

$$\boldsymbol{u},_{\alpha}=\gamma_{\alpha\lambda}\boldsymbol{a}^{\lambda}+\boldsymbol{\phi}\times\boldsymbol{a}_{\alpha}\ . \tag{23}$$

Differentiating (10), one obtains

$$\begin{aligned}
&\boldsymbol{\phi},_{\alpha}=\varepsilon^{\lambda\mu}\mu_{\alpha\lambda}\boldsymbol{a}_{\mu}+\zeta_{\alpha}\boldsymbol{n}\ ,\\
&\mu_{\alpha\lambda}=-\varphi_{\lambda|\alpha}-\varepsilon_{\lambda\rho}b_{\alpha}^{\rho}\phi\ ,\quad \zeta_{\alpha}=\phi,_{\alpha}+\varepsilon^{\rho\lambda}\varphi_{\lambda}b_{\alpha\rho}\ .
\end{aligned} \tag{24}$$

Then

$$\begin{aligned}
&\varepsilon^{\alpha\beta}\boldsymbol{u}_{|\alpha\beta}=\left(\varepsilon^{\alpha\beta}\gamma_{\alpha\lambda|\beta}+\zeta_{\lambda}\right)\boldsymbol{a}^{\lambda}+\varepsilon^{\alpha\beta}\left(\gamma_{\alpha\lambda}b_{\beta}^{\lambda}+\mu_{\alpha\beta}\right)\boldsymbol{n}=\boldsymbol{0}\ ,\\
&\varepsilon^{\alpha\beta}\boldsymbol{\phi}_{|\alpha\beta}=\varepsilon^{\alpha\beta}\left(\varepsilon^{\rho\lambda}\mu_{\alpha\rho|\beta}+b_{\alpha}^{\lambda}\zeta_{\beta}\right)\boldsymbol{a}_{\lambda}+\varepsilon^{\alpha\beta}\left(\zeta_{\alpha|\beta}+\varepsilon^{\lambda\rho}\mu_{\alpha\lambda}b_{\beta\rho}\right)\boldsymbol{n}=\boldsymbol{0}\ .
\end{aligned} \tag{25}$$

Using the relations (3) and (4), after some transformations one obtains

$$\mu_{\alpha\beta} = \kappa_{\alpha\beta} + b_\alpha^\lambda \gamma_{\beta\lambda} \, . \tag{26}$$

This indicates that the second expression of $(25)_1$ identically vanishes. Then one can solve the first expression in $(25)_1$ for ζ_λ and introduce the result into $(25)_2$. By changing some indices, the remaining three compatibility conditions become

$$\begin{aligned} &\left(\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\mu_{\rho\lambda}\right)|_\alpha - b_\lambda^\beta \left(-\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\gamma_{\rho\lambda}\right)|_\alpha = 0 \, , \\ &\left(-\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\gamma_{\rho\lambda}\right)|_{\alpha\beta} + b_{\alpha\beta}\left(\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\mu_{\rho\lambda}\right) = 0 \, . \end{aligned} \tag{27}$$

Static-Geometric Analogy and Complex Shell Relations

Between the equilibrium equations (18) and the compatibility conditions (27) there exists the following correspondence:

$$T^{\alpha\beta} \Leftrightarrow \varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\mu_{\rho\lambda} \, , \quad M^{\alpha\beta} \Leftrightarrow -\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\gamma_{\rho\lambda} \, . \tag{28}$$

When the surface stress measures in (18) are replaced by the surface deformation measures according to (28), the homogeneous equilibrium equations (18) are converted into the compatibility conditions (27). The correspondence is known as the static-geometric analogy in the linear theory of thin shells.

The analogy (28) allows one to introduce three stress functions $\bar{u}_\alpha, \bar{w}$ by the relations

$$T^{\alpha\beta} = T_*^{\alpha\beta} + Ehc\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\bar{\mu}_{\rho\lambda} \, , \quad M^{\alpha\beta} = M_*^{\alpha\beta} - Ehc\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\bar{\gamma}_{\rho\lambda} \, , \quad c = \frac{h}{\sqrt{12(1-\nu^2)}} \, . \tag{29}$$

Here $T_*^{\alpha\beta}$ and $M_*^{\alpha\beta}$ are some particular solutions of the inhomogeneous equilibrium equations (18), and the expressions $\bar{\gamma}_{\rho\lambda}$ and $\bar{\mu}_{\rho\lambda}$ are similar to $(4)_1$ and (26), respectively, only now constructed by corresponding stress functions $\bar{u}_\alpha, \bar{w}$.

With (29) one can introduce the surface complex stress measures

$$\tilde{T}^{\alpha\beta} = T_*^{\alpha\beta} + i\,Ehc\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\tilde{\mu}_{\rho\lambda} \, , \quad \tilde{M}^{\alpha\beta} = M_*^{\alpha\beta} - i\,Ehc\varepsilon^{\alpha\rho}\varepsilon^{\beta\lambda}\tilde{\gamma}_{\rho\lambda} \, , \quad i = \sqrt{-1} \, , \tag{30}$$

where $\tilde{\gamma}_{\rho\lambda}$ and $\tilde{\mu}_{\rho\lambda}$ are expressions similar to $(4)_1$ and (26) only now constructed by the complex translations $\tilde{u}_\alpha = u_\alpha + i\bar{u}_\alpha$, $\tilde{w} = w + i\,\bar{w}$. When the compatibility conditions (27) are multiplied by $i\,Ehc$ and added with the corresponding equilibrium equations (18), this gives the following set of three equations for the complex stress measures:

$$\tilde{T}^{\alpha\beta}|_\alpha - b_\lambda^\beta \tilde{M}^{\alpha\lambda}|_\alpha + f^\beta = 0 \, , \quad \tilde{M}^{\alpha\beta}|_{\alpha\beta} + b_{\alpha\beta}\tilde{T}^{\alpha\beta} + f = 0 \, . \tag{31}$$

When expressed in terms of complex translations, the above system of PDEs for the complex independent variables is of the 4th order to be solved in the complex domain, while the system (18) of PDEs for the real translations is of the 8th order in the real domain.

The complex formulation of the linear thin shell theory was used to solve analytically a number of linear shell problems presented, for example, in the books by Novozhilov (1964), Chernykh (1964), and Novozhilov et al. (1991).

Constitutive Equations

When the shell is made of an elastic material, the principle (5) requires the existence of the surface strain energy density $\Sigma\left(\gamma_{\alpha\beta},\kappa_{\alpha\beta}\right)$, per unit area of $\mathcal{M}$, such that $N^{\alpha\beta}=\partial\Sigma/\partial\gamma_{\alpha\beta}$ and $M^{\alpha\beta}=\partial\Sigma/\partial\kappa_{\alpha\beta}$. The explicit expression for Σ depends on the shell material properties, but also on the undeformed shell geometry: its thickness, curvatures of $\mathcal{M}$ and the internal structure across the thickness.

In case of a homogeneous isotropic shell undergoing small elastic strains, already Love (1888, 1927) used $\Sigma\left(\gamma_{\alpha\beta},\kappa_{\alpha\beta}\right)$ consisting of the sum of two quadratic functions describing stretching and bending energies of the shell. The error of such an approximation was analysed in Novozhilov and Finkel'stein (1943) and in several later papers. In particular, according to Koiter (1960) the consistently approximated strain energy density is given by

$$\begin{aligned}\Sigma&=\frac{h}{2}H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu}+\frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}\right)+O\left(Eh\eta^2\theta^2\right),\\ H^{\alpha\beta\lambda\mu}&=\frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu}+a^{\alpha\mu}a^{\beta\lambda}+\frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right).\end{aligned}\tag{32}$$

Here E is the Young modulus and ν is the Poisson ratio of the linearly elastic isotropic material, while the small parameter θ is defined as $\theta=\max\left(h/L,\sqrt{h/R},\sqrt{\eta}\right)$, where L is the smallest length of geometric and deformation patterns on $\mathcal{M}$. The form $(32)_1$ of $\Sigma\left(\gamma_{\alpha\beta},\kappa_{\alpha\beta}\right)$ was subsequently justified by asymptotic analysis of the equations of 3D linearized elasticity as indeed the consistent first approximation to the 3D strain energy density of the shell (see review by Ciarlet 2000). The material tensor $H^{\alpha\beta\lambda\mu}$ in $(32)_2$ corresponds to the plane stress state in the shell space in accordance with the assumption 2 indicated in Introduction.

With $(32)_1$, the constitutive equations of isotropic elastic shells are given by

$$\begin{aligned}N^{\alpha\beta}&=\frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}}=\frac{Eh}{1-\nu^2}\left[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma_\kappa^\kappa\right]+O\left(Eh\eta\theta^2\right),\\ M^{\alpha\beta}&=\frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}}=\frac{Eh^3}{12(1-\nu^2)}\left[(1-\nu)\kappa^{\alpha\beta}+\nu a^{\alpha\beta}\kappa_\kappa^\kappa\right]+O\left(Eh^2\eta\theta^2\right).\end{aligned}\tag{33}$$

Summarizing, the boundary value equilibrium problem of a thin isotropic elastic shell can be formulated in terms of three translations u_α, w of $\mathcal{M}$ as the basic independent field variables. The BVP consists of three scalar equilibrium equations (18) with (19), the four natural static (20) and $(15)_2$ and/or four work-conjugate kinematic boundary conditions (17), the constitutive equations (33), and the kinematic relations (4).

The error indicated in $(32)_1$ suggests that within the accuracy of the first approximation to the strain energy function $(33)_1$ one can apply various alternative definitions of the bending tensor, provided that they differ from $\kappa_{\alpha\beta}$ by terms of the type $b_\alpha^\lambda\gamma_{\lambda\beta}$. In particular, Koiter (1960) used the bending tensor $\rho_{\alpha\beta}=-\kappa_{\alpha\beta}-1/2\left(b_\alpha^\lambda\gamma_{\lambda\beta}+b_\beta^\lambda\gamma_{\lambda\alpha}\right)$ and Budiansky and Sanders (1963) by several additional criteria found it to be the "best" bending tensor for the linear theory of thin isotropic elastic shells. However, some shell relations compatible with $\rho_{\alpha\beta}$ become more complex and less convenient in general discussions.

According to (26), the non-symmetric tensor $\mu_{\alpha\beta}$ may also be used as the bending tensor of the linear theory of thin elastic shells. In this case the constitutive equations $(33)_2$ of an isotropic elastic shell are given by

$$M^{\alpha\beta} = \frac{\partial \Sigma}{\partial \mu_{\alpha\beta}} = \frac{Eh^3}{12(1-\nu^2)}\left[(1-\nu)\mu^{\alpha\beta} + \nu a^{\alpha\beta}\mu^{\kappa}_{\cdot\kappa}\right] + O\left(Eh^2\eta\theta^2\right). \quad (34)$$

Conclusions

The limited space for this entry does not allow one to discuss here many other important problems of the linear theory of thin isotropic elastic shells. The literature in the field is numerous and some early important contributions are inaccessible through Internet. The interested reader should consult references given in the books referred to below.

Cross References

Surface Geometry, Elements
Thin Elastic Shells, Lagrangian Geometrically Non-linear Theory
Junctions in Irregular Shell Structures
Elastic Shells, Resultant Non-linear Theory

References

Başar Y, Krätzig WB (1985) Mechanik der Flächentragwerke. F. Vieveg & Sohn, Braunschweig

Başar Y, Krätzig WB (2001) Theory of shell structures, 2nd ed. VDI Verlag, Düsseldorf

Budiansky B, Sanderrs JL (1963) On the “best” first-order linear shell theory. In: Progress in Applied Mechanics (Prager Anniv Vol). Macmillan, New York, pp 126-140

Chernykh KF (1964) Linear theory of shells, Part 2 (in Russian). University Press, Leningrad. English translation: NASA-TT-F-II 562, 1968

Ciarlet PG (2000) Mathematical elasticity, Vol. III: Theory of shells. Elsevier, Amsterdam et al.

Flügge W (1972) Tensor analysis and continuum mechanics. Springer-Verlag, Berlin et al.

Goldenveizer AL (1961) Theory of elastic thin shells. Pergamon press, Oxford

Green AE, Zerna W (1954) Theoretical elasticity. Clarendon Press, Oxford

Koiter WT (1961) A consistent first approximation in the general theory of thin elastic shells. In: Proc IUTAM Symp on the Theory of Thin Elastic Shells. North-Holland, Amsterdam, pp 139-154

Love AEH (1888) The small free vibrations of a thin elastic shells. Phil. Trans. Royal Soc. Lindon A 179:491-546

Love AEH (1927) A treatise on the mathematical theory of elasticity, 4th edn. University Press, Cambridge

Lurie AI (1940) General theory of elastic thin shells (in Russian). Prikl Matem Mekh 4(2):7-34

Naghdi PM (1963) Foundations of elastic shell theory. In: Sneddon IN, Hill R (eds) Progress in Solid Mechanics IV. North-Holland, Amsterdam, pp 1-90

Novozhilov VV (1964) Thin shell theory. P. Noordhoff, Groningen

Novozhilov VV, Chernykh KF, Mikhailovskii EI (1991) The linear theory of thin shells (in Russian). Politekhnika, Leningrad

Novozhilov VV, Finkel'shtein PM (1943) On the error of Kirchhoff's hypotheses in shell theory (in Russian). Prikl Matem Mekh 7(5): 331-340

Pietraszkiewicz W (1980) Finite rotations in the nonlinear theory of thin shells. In: Olszak W (ed) Thin Shell Theory: New Trends and Applications. CISM Courses and Lectures No 240. Springer-Verlag, Wien-New York, pp 154-208

Pietraszkiewicz W (2017) Surface geometry, Elements. In: Altenbach H, Őchsner A (eds) Encyclopedia of Continuum Mechanics. Springer, Berlin 2018